目　　　录

第 1 章 绪 论

1．1 题

1）承受总纵弯曲构件：

连续上甲板，船底板，甲板及船底纵骨，连续纵桁，龙骨等远离中和轴的纵向连续构件（舷侧列板等）

2）承受横弯曲构件：甲板强横梁，船底肋板，肋骨

3）承受局部弯曲构件：甲板板，平台甲板，船底板，纵骨等

4）承受局部弯曲和总纵弯曲构件：甲板，船底板，纵骨，递纵桁，龙骨等

1．2 题

甲板板：纵横力（总纵弯曲应力沿纵向，横向货物或上浪水压力，横向作用）

舷侧外板：横向水压力等骨架限制力沿中面

内底板：主要承受横向力货物重量，骨架限制力沿中面为纵向力

舱壁板：主要为横向力如水，货压力也有中面力

第 2 章 单跨梁的弯曲理论

2．1 题

设坐标原点在左跨时与在跨中时的挠曲线分别为 v(x)与 v(x_1)

1）图 2.1° $$v(x)=\frac{M_0x^2}{2EI}+\frac{N_\circ x^3}{6EI}+\Big\|_{l/4}\frac{p(x-l/4)^3}{6EI}+\Big\|_{l/2}\frac{p(x-l/2)^3}{6EI}+\Big\|_{3l/4}\frac{p(x-3l/4)^3}{6EI}$$

原点在跨中： $$v_1(x_1)=v_0+\frac{M_0x_1^2}{2EI}+\frac{N_\circ x_1^3}{6EI}+\Big\|_{l/4}\frac{p(x-l/4)^3}{6EI}$$ ，

$$\begin{cases} v_1(l/2)=0 \quad v_1'(l/2)=0 \\ v_1'(0)=0 \quad N_1(0)=p/2 \end{cases}$$

2）图2.2° $$v(x)=\theta_0x+\frac{Mx^2}{2EI}+\frac{N_\circ x^3}{6EI}+\Big\|_{l/3}\frac{p(x-l/3)^3}{6EI}$$

3）图2.3° $$v(x_x)=\theta_0x_x+\frac{N_\circ x^3}{6EI}+\int_0^x\frac{qx^3dx}{6EI}-\Big\|_{l/2}\frac{p(x-l/2)^3}{6EI}$$

2．2 题

a) $$v_1=v_{pp}+v_p=\frac{pl^3}{6EI}\left[\frac{1}{16}(3\times\frac{1}{4}\times\frac{3}{4}-\frac{1}{4})\right]+\frac{pl^3}{6EI}\left[\frac{1}{4}\times\frac{1}{16}(\frac{3}{2}-2\times\frac{1}{4})\right]$$

$$=pl^3 \big/ 512EI$$

$$V_2 = pl^3 \big/ 6EI\left[\frac{1}{4}\left(\frac{9}{16}-\frac{1}{2}\right)+(\frac{1}{4})^3\right]+pl^3 \big/ 192EI = pl^3 \big/ 96EI$$

b) $$v'(0)=\frac{-Ml}{3EI}+\frac{Ml}{6EI}+\frac{2Pl^2/9}{6EI}(1+2\big/3)$$

$$=-\frac{0.1Pl^2}{6EI}+\frac{5Pl^2}{3\times 27EI}=73Pl^2\big/1620EI$$

$$\theta(l)=\frac{-Ml}{3EI}+\frac{Ml}{6EI}-\frac{2Pl^2/9}{6EI}(1+1\big/3)$$

$$=-\frac{0.1Pl^2}{6EI}-\frac{4Pl^2}{3\times 27EI}=-107Pl^2\big/1620EI$$

$$v\left(l\big/3\right)=\frac{p\left(\frac{l}{3}\right)^2\left(\frac{2l}{3}\right)^2}{3EIl}-\frac{l^2\big/3}{6EI}\left(1-\frac{1}{3}\right)\left[m\left(2-1\big/3\right)-m\left(1+1\big/3\right)\right]$$

$$=37pl^2\big/2430EI$$

c) $$v\left(l\big/2\right)=\frac{ql^4}{192EI}-\frac{7\big/3\,ql^4}{768EI}=\frac{5ql^4}{2304EI}$$

$$v'(0)=\frac{ql^3}{24EI}-\frac{pl^2}{16EI}-\frac{\left(ql^2\big/16\right)l}{6EI}=\frac{ql^3}{8EI}\left[\frac{1}{3}-\frac{1}{6}-\frac{1}{12}\right]=ql^3\big/96EI$$

d) 2.1°图、2.2°图和2.3°图的弯矩图与剪力图如图 2.1、图 2.2 和图 2.3

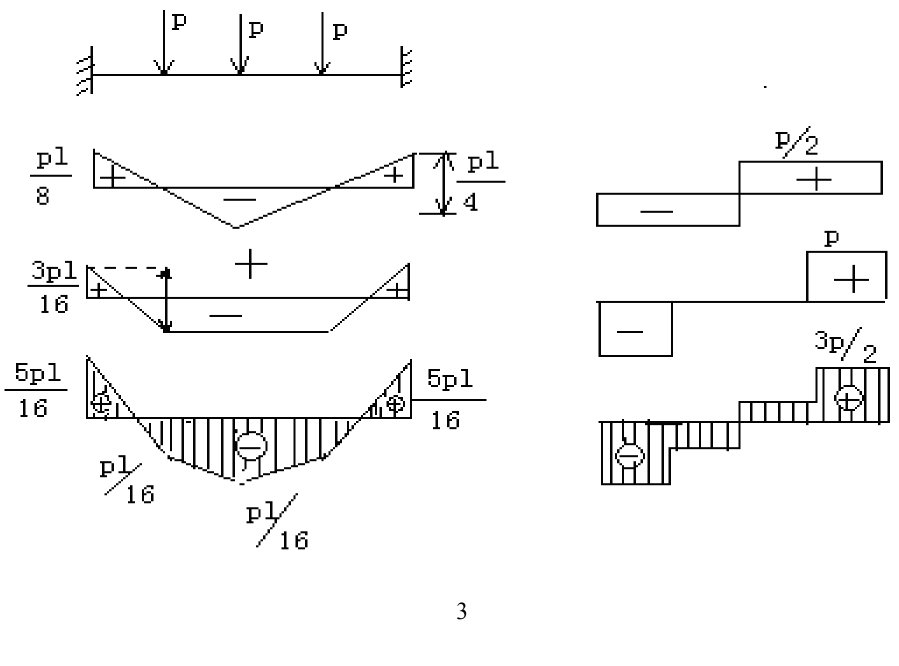


图 2.1

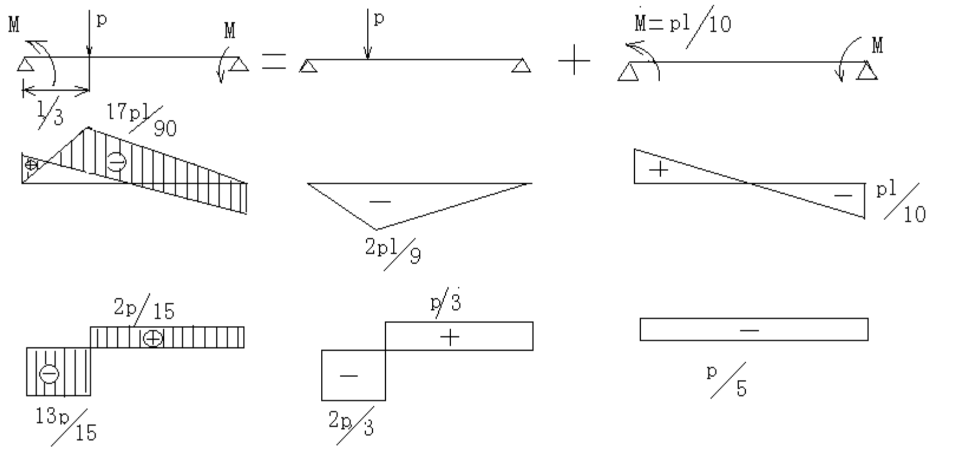


图 2.2

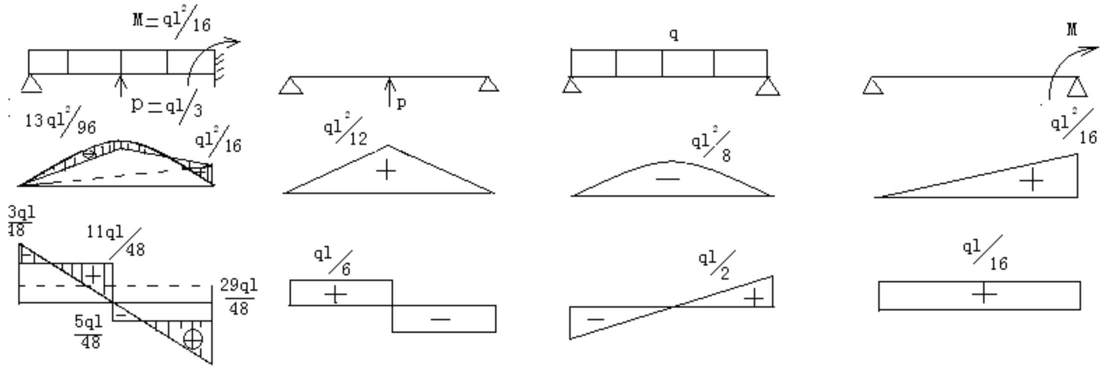


图 2.3

2.3 题

1）

$$\because \theta_{右}=\frac{Ml}{6EI}-\frac{ql^3}{24EI}-\frac{2l^2}{45EI}\left[\frac{l}{2}(q_2-q_1)\right]+\frac{\bar{M}l}{3EI}=0$$

$$\therefore \bar{M}={13q_1l^2}\Big/{120}$$

2）$$\theta_0=-\frac{Ml}{3EI}+\frac{q_1l^3}{24EI}+\frac{7l^2}{180EI}\left[\frac{l}{2}lq_1\right]-\frac{\bar{M}l}{6EI}$$

$$=\frac{q_1l^3}{EI}\left(-\frac{1}{18}+\frac{1}{24}+\frac{7}{360}-\frac{13}{6\times120}\right)=-\frac{q_1l^3}{80EI}$$

2.4 题

图2.5° $\because v(x)=v_0+\theta_0x+{N_0x^3}\Big/{6EI}$， $v_0=A(p-N_0)$

$$\therefore v(x)=Ap+\theta_0x+\left(\frac{x^3}{6EI}-A\right)N_0$$

如图 2.4， 由$v(l)=v'(l)=0$得

$$\left.\begin{aligned}&Ap+\theta_0 l+\left(\frac{l^3}{6EI}-A\right)N_0=0\\&\theta_0+\frac{l^2}{2EI}N_0=0\end{aligned}\right\}\text{解出}$$

$$\begin{cases}\theta_0=\dfrac{-Ap}{l}=-\dfrac{pl^2}{6EI}\\N_0=p/3\end{cases}$$

$$\therefore v(x)=\frac{pl^3}{9EI}\left(1-\frac{3x}{2l}+\frac{x^3}{2l^3}\right)$$

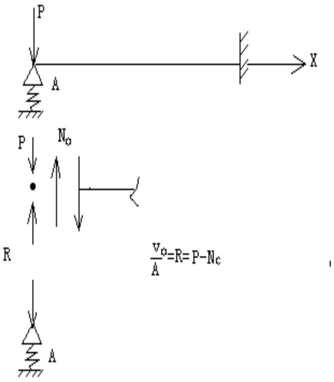


图 2.4

图2.6°

$$v(x)=\theta_1 x+\frac{M_0 x^2}{2EI}+\frac{N_0 x^3}{6EI}$$

由$v(l)=0,\quad v'(l)=\theta_2$ 得

$$\left.\begin{aligned}&\theta_1 l+\frac{M_0 l^2}{2EI}+\frac{N_0 l^3}{6EI}=0\\&\theta_1+\frac{M_0 l}{EI}+\frac{N_0 l^2}{2EI}=\theta_2\end{aligned}\right\}\text{解得}\quad\begin{cases}M_0=-\dfrac{4EI}{l}\theta_1-\dfrac{2EI}{l}\theta_2\\N_0=\dfrac{6EI}{l^2}(\theta_1+\theta_2)\end{cases}$$

$$\therefore v(x)=\theta_1 x+\frac{(2\theta_1+\theta_2)x^2}{l}+\frac{(\theta_1+\theta_2)x^3}{l^2}$$

2.5 题

图2.5°:（剪力弯矩图如２． ５）

$$\therefore R_1=\frac{pl-\bar{M}}{l}=p-p/3=2p/3$$

$$v_0=AR=\frac{l^3}{6EI}.2p/3=pl^2/9EI$$

$$v\left(\frac{l}{2}\right)=\frac{v_0}{2}-\frac{\bar{M}l^2}{16EI}=\frac{pl^3}{18EI}-\frac{pl^3}{48EI}=\frac{5pl^3}{144EI}$$

$$v'(0)=\theta_0=-\frac{v_0}{l}-\frac{\bar{M}l}{6EI}=\frac{-pl^2}{9EI}-\frac{pl^2}{18EI}=-\frac{pl^2}{6EI}$$

$$\bar{M}=\frac{pa}{K_A}\left[\bar{A}+\frac{b}{6l}\left(1+b/l\right)\right]\ ,$$

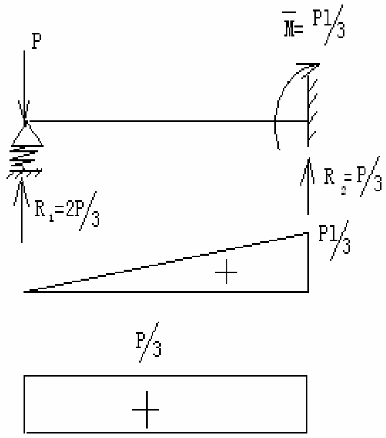


图 2.5

将 $a=l,b=0\quad\bar{A}=l/6,\quad K_A=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$ 代入得：$\bar{M}=\frac{pl}{1/2}\left(1/6\right)=pl/3$

2.7°图：（剪力弯矩图如２． ６）

$$v_1 = A_1R_1 = \frac{0.05l^3}{EI}\cdot\frac{ql}{2} = \frac{ql^4}{40EI}$$

$$v_2 = A_2R_2 = \frac{l^3}{50EI}\cdot\frac{ql}{2} = \frac{ql^4}{100EI}$$

$$v\left(\frac{l}{2}\right) = \frac{5ql^4}{384EI} + \frac{ql^4}{2EI}\left(\frac{1}{40}+\frac{1}{100}\right)$$

$$= \frac{ql^4}{EI}\left(\frac{5}{384}+\frac{7}{400}\right) = \frac{293ql^4}{9600EI}$$

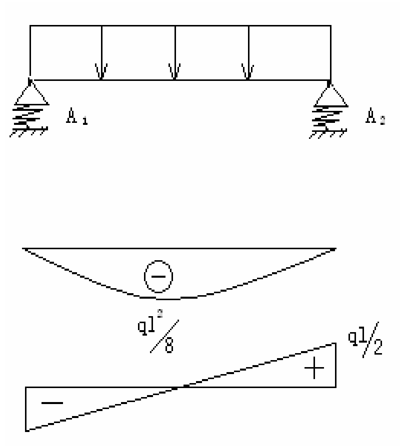


图２． ６

$$\theta(0) = \frac{ql^3}{24EI} - \frac{v_1 - v_2}{l} = \frac{ql^3}{EI}\left(\frac{1}{24}-\frac{1}{40}+\frac{1}{100}\right) = \frac{2ql^3}{75EI}$$

$$\theta(l) = -\frac{ql^3}{24EI} - \frac{v_1 - v_2}{l} = \frac{ql^3}{EI}\left(-\frac{1}{24}-\frac{1}{40}+\frac{1}{100}\right) = \frac{-17ql^3}{300EI}$$

图2.8°（剪力弯矩图如２． ７）

$$\bar{M} = \frac{Qa}{24}\cdot\frac{1}{K_A}\left[12\bar{A}+\left(1+\frac{b}{l}\right)^2\right]$$

由$Q = qa, a = l, b = 0,$

$\bar{\alpha} = 1/8, \bar{A} = 1/24$

$K_A = 1/8 + 1/24 + 1/3 = 1/2$, 代入得

$$\bar{M} = \frac{ql^2}{24}\times 2\times\left(12\times 1/24 + 1\right) = ql^2/8$$

$$R_1 = \frac{ql}{2} - \frac{ql}{8} = 3ql/8,$$

$$v_0 = AR_1 = \frac{ql^4}{64EI}$$

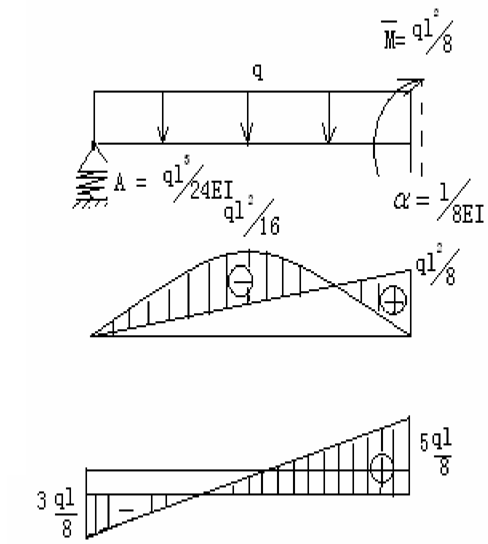


图２． ７

$$\therefore v\left(\frac{l}{2}\right) = \frac{5ql^4}{384EI} + \frac{ql^4}{128EI} - \frac{\bar{M}l^2}{16EI} = \frac{5ql^4}{384EI}$$

$$\theta(0) = \frac{ql^3}{24EI} - \frac{v_0}{l} - \frac{\bar{M}l}{6EI} = \frac{ql^3}{EI}\left(\frac{1}{24}-\frac{1}{64}-\frac{1}{48}\right)$$

$$= \frac{ql^3}{192EI}$$

$$\theta(l) = -\alpha\bar{M} = -\frac{l}{8EI}\cdot\frac{ql^2}{8} = -ql^3/64EI$$

2.6 题

$$dv_2=\gamma_{\max}.dx=\frac{\tau_{\max}}{G}dx=-\frac{N}{GA_s}dx$$

$$v_2=\int\frac{N}{GA_s}dx\xrightarrow{N=EIv_1'''}-\frac{EI}{GA_s}v_1''+C_1$$

$$\therefore v=v_1+v_2=\left[f(x)+ax^3\Big/6+bx^2\Big/2+cx+d\right]-\frac{EI}{GA_s}\left[f''(x)+ax+b\right]+C_1$$

$$=f(x)-\frac{EI}{GA_s}f''(x)+ax^3\Big/6+bx^2\Big/2+\left(c-\frac{EI}{GA_s}a\right)x+d_1$$

式中 $f(x)=qx^4\Big/24EI$ $f''(x)=qx^2\Big/2EI$

由于 $v(0)=v_1''(0)=0$ 可得出 $d_1=b=0$

由 $v(l)=v_1''(l)=0$ 得方程组：

$$\begin{cases}\dfrac{ql^4}{24EI}-\dfrac{EI}{GA_s}\dfrac{ql^2}{2EI}+\dfrac{al^3}{6}+\left(c-\dfrac{EI}{GA_s}a\right)l=0\\[2ex]\dfrac{ql^2}{2EI}+al=0\end{cases}$$

解出：a=$\dfrac{ql}{2EI}$， $c=\dfrac{ql^3}{24EI}$

$$\therefore v(x)=\frac{qx^4}{24EI}-\frac{qlx^3}{12EI}-\frac{qx^2}{2GA_s}+\left(\frac{qx^3}{24EI}+\frac{ql}{2GA_s}\right)x$$

$$\therefore v(\frac{l}{2})=\frac{5ql^4}{384EI}+\frac{ql^2}{8GA_s}$$

2.7.题

先推广到两端有位移$\Delta_i,\theta_i,\Delta_j,\theta_j$情形：$\left(令\Delta=\Delta_i-\Delta_j,\beta=\dfrac{12EI}{GA_sl^2}\right)$

$$\because v=ax^3\Big/6+bx^2\Big/2+cx+d_1-\frac{EI}{GA_s}ax$$

$$\left.\begin{array}{l}而v_0=\Delta_i\quad\therefore d_1=v(0)=\Delta_i\\ 由v_1'(0)=\theta_i\quad\therefore c=\theta_i\end{array}\right\}$$

$$\left.\begin{array}{l}由v(l)=\Delta_j\quad\therefore\dfrac{al^3}{6}+\dfrac{bl^2}{2}+\theta_il+\Delta_i-\dfrac{EI}{GA_s}al=\Delta_j\\[2ex] 由v_1'(l)=\theta_j\quad\therefore\dfrac{al^2}{2}+bl=\theta_j\end{array}\right\}$$

$$\text{解出}\begin{cases} a=\dfrac{\theta}{l^2(1+\beta)}\left[\theta_i+\theta_j-2\Delta/l\right] \\ b=\dfrac{\theta_j-\theta_i}{l}-\dfrac{3}{l}\dfrac{1}{(1+\beta)}\left(\theta_i+\theta_j-2\Delta/l\right) \end{cases}$$

$$\therefore M(0)=EIv_1''(0)=EIb=\frac{EI}{l(1+\beta)}\left[6\frac{\Delta}{l}+(\beta-2)\theta_j-(\beta+4)\theta_i\right]$$

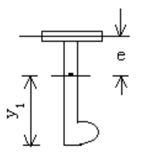
$$=-\frac{EI}{l(1+\beta)}\left[\frac{6}{l}\Delta_i-\frac{6}{l}\Delta_j+(\beta+4)\theta_i+(-\beta+2)\theta_j\right]$$

$$\begin{cases} N(0)=EIv_1'''(0)=EIa=\dfrac{6EI}{l^2(1+\beta)}\left[\theta_i+\theta_j-\dfrac{2}{l}(\Delta_j-\Delta_i)\right] \\ N(l)=N(0) \\ M(l)=EIv_1''(l)=EI(b+al)=\dfrac{EI}{l(1+\beta)}\left[(\beta+4)\theta_j+(2-\beta)\theta_i-6\dfrac{\Delta}{l}\right] \end{cases}$$

令上述结果中$\Delta_i=0$，即$\Delta=\Delta_j$同书中特例

2.8 题 已知：$l=3\times75=225cm,\quad t=1.8cm,\quad s=75cm\quad \sigma_0=1050\,{kg}/{cm^2}$

$$q=\gamma hs=1025\times10\times0.75=76.875\,{kg}/{cm}$$

	面积 cm^2	距参考轴 cm	面积距 cm^3	惯性矩 cm^4	自惯性矩 cm^4
外板1.8×45	81	0	0	0	(21.87) 略
球扁钢 N_Q24a	38.75			9430.2	2232
$\sum$	119.8	15.6	604.5	9430.2	2253.9
	A		B	C=11662	

$$e={B}/{A}=5.04cm\quad I=C-{B^2}/{A}=11662-{604.5^2}/{119.8}=8610cm^4$$

计算外力时面积$A=75\times1.8+38.75=174cm^2$

计算I时，带板$be=\min\left\{\frac{l}{5},s\right\}=\frac{l}{5}=45cm$

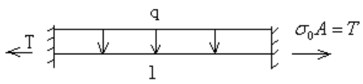
1）.计算组合剖面要素：

形心至球心表面$y_1=h+{t}/{2}-e=24+0.9-5.04=19.86cm$形心至最外板纤维

$$y_2 = e + t/2 = 5.94cm \therefore w_1 = I/y_1 = 8610/19.86 = 433.5cm^3$$

$$w_2 = I/y_2 = 8610/5.94 = 1449.4cm^3$$

$$u = \frac{l}{2}\sqrt{\frac{\sigma_0 A}{EI}} = \frac{225}{2}\sqrt{\frac{1050\times174}{2\times10^6\times8610}} = 0.366$$



$$x(u) = 0.988, \quad \varphi_1(u) = 0.980$$

$$\bar{M} = \frac{ql^2}{12}x(u) = \frac{76.875}{12}\times225^2\times0.988 = 320424(kg.cm)$$

$$M_{中} = -\frac{ql^2}{24}\varphi_1(u) = -\frac{1}{24}\times76.875\times225^2\times0.980 = -158915(kgcm)$$

$$\left.\begin{aligned}
\sigma_{中}^{球头} &= \sigma_0 + \frac{|M_{中}|}{w_1} = 1050 + \frac{158915}{433.5} = 1416\, kg/cm^2 \\
\sigma_{固端}^{板} &= \sigma_0 + \frac{\bar{M}}{w_2} = 1050 + \frac{320424}{1450} = 1271\, kg/cm^2 \\
\sigma_{端}^{球头} &= \sigma_0 + \frac{\bar{M}}{w_1} = 1050 + \frac{320424}{433.5} = 378\, kg/cm^2
\end{aligned}\right\} \therefore \sigma_{\max} = 1416\, kg/cm^2$$

若不计轴向力影响，则令 u=0 重复上述计算：

$$\sigma_{\max} = \sigma_{中}^{球头} = \sigma_0 + \frac{ql^2}{24}\Big/ w_1 = 1050 + \frac{76.875\times225^2}{24\times433.5} = 1424\, kg/cm^2$$

相对误差：$\dfrac{|1424-1416|}{1424} = 0.56\%$

结论：轴向力对弯曲应力的影响可忽略不及计。结果是偏安全的。

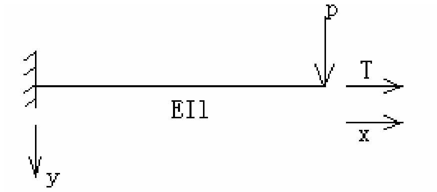
2.9.题

$$\because EIv^{IV} - Tv'' = 0, EIv''' = N + Tv'$$

$$\therefore v^{IV} - \sqrt{\frac{T}{EI}}^2 V'' = 0, v^{IV} - K^2 V'' = 0 \text{式中} k = \sqrt{\frac{T}{EI}}$$

特征根：$r_{1,2} = 0, \quad r_3 = k \quad r_4 = -k$

$$\therefore v = A_1 + A_2 kx + A_3 chkx + A_4 shkx$$



$$\because \left.\begin{aligned} v(0) &= 0 \\ v'(0) &= 0 \end{aligned}\right\} \quad \therefore \left.\begin{aligned} A_1 + A_3 &= 0 \\ A_2 + A_4 &= 0 \end{aligned}\right\}$$

$$\because \begin{cases} v''(l) = 0 \\ EIv'''(l) = N(l) + Tv'(l) \end{cases}$$

$$\therefore \begin{cases} A_3 chkl + A_4 shkl = 0 \\ EIk^3\left(A_3 shkl + A_4 chkl\right) = -p + Tk\left(A_2 + A_3 shkl + A_4 chkl\right) \end{cases}$$

解得：

$$A_1 = -\frac{p}{kT} thkl, A_2 = \frac{p}{kT}, A_3 = \frac{p}{kT} thkl, A_4 = -\frac{p}{kT}$$

$$\therefore v(x) = -\frac{p}{kT}\left(thkl - kx - thklchkx + shkx\right)$$

$$= -\frac{p}{EIk^3}\left[thkl\left(1 - chkx\right) + \left(shkx - kx\right)\right]$$

2.10 题

$$EIv^{IV} + T^* v'' = 0 \quad \left(EIv^{IV} = N - T^* v'\right)$$

$$v^{IV} + k^{*2} v'' = 0 \text{式中} k^* = \sqrt{T^*/EI}$$

特征方程：$r^4 + k^2 r^2 = 0$

特征根：$r_{1,2} = 0, \quad r_3 = ik^* \quad r_4 = -ik^*$

$$\therefore v = A_1 + A_2 k^* x + A_3 \sin k^* x + A_4 \cos k^* x$$

$$\because \left.\begin{matrix} v(0) = 0 \\ EIv''(0) = m \end{matrix}\right\} \quad \therefore \left.\begin{matrix} A_1 + A_4 = 0 \\ -A_4 k^{*2} = m/EI \end{matrix}\right\}$$

$$\because \begin{cases} v''(l) = 0 \\ EIv'''(l) = -T^* v'(l) \end{cases}$$

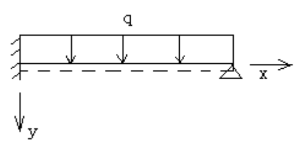
$$\therefore \begin{cases} A_3 \sin k^* l + A_4 \cos k^* l = 0 \\ -k^{*3}\left(A_3 \cos k^* l - A_4 \sin k^* l\right) = -k^{*3}\left(A_2 + A_3 \cos k^* l - A_4 \sin k^* l\right) \end{cases}$$

解得：$A_3 = -\frac{m}{T^*} ctgk^* l, A_2 = 0$

$$\therefore v'(0) = \left[A_2 k^* + A_3 k^* \cos k^* - A_4 k^* \sin k^*\right]_{x=0} = A_3 k^* = \frac{m}{k^* EItgk^* l}$$

2.11 题

图２．１２^0



由$v'(0)=0$　协调条件查附录图：

令 A=0 ∴B=0 u=$\frac{l}{2}\sqrt[4]{\frac{k}{EI}}=\frac{l}{2}\sqrt[4]{\frac{64EI}{4EIl}}=1$

$$\frac{ql^3}{24EI}\psi_2(u)-\frac{\bar{M}l}{3EI}\psi_0(u)=0$$

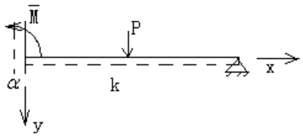
$$\bar{M}=\frac{ql^2}{8}\frac{\psi_2(u)}{\psi_0(u)}=\frac{ql^2}{8}\frac{.609}{0.752}=0.101ql^2$$

$$v\left(\frac{l}{2}\right)=\frac{q}{k}\left[1-\frac{\psi_0(u)}{1+B}\right]+\frac{\bar{M}}{2\alpha^2EI}\left[\frac{v_1(2u)v_3\left(\alpha\frac{l}{2}\right)-v_3(2u)v_1\left(\alpha\frac{l}{2}\right)}{v_1^2(2u)+v_3^2(2u)}\right]$$

$$\xrightarrow[u=1\quad B=0]{\alpha=2/l}=\frac{ql^4}{64EI}(1-0.448)+\frac{0.101ql^4}{8EI}\left(\frac{\frac{1.9115}{\sqrt{2}}\cdot\frac{0.6635}{\sqrt{2}}-\frac{4.9301}{\sqrt{2}}\cdot\frac{1.9335}{\sqrt{2}}}{1.9115^2/2-4.9301^2/2}\right)$$

$$=0.0049ql^4/EI$$

2.13°图



$$\theta(0)=\frac{pl^2}{16EI}x_0(u)-\frac{\bar{M}l}{3EI}\psi_0(u)=\alpha\bar{M}$$

$$\bar{M}=\frac{pl^2}{16EI}x_0(u)\Big/\left[\alpha+\frac{l\psi_0(u)}{3EI}\right]$$

将$u=1,\alpha=l/12EI$代入得：

$$\bar{M}=\frac{pl}{16}\times0.591\Big/\left(\frac{1}{12}-\frac{0.72}{3}\right)=0.111pl$$

$$v\left(\frac{l}{2}\right)=\frac{pl^3}{48EI}\psi_2(u)+\frac{\bar{M}}{2\alpha^2EI}\cdot\left[\frac{v_1(2u)v_3\left(\alpha\frac{l}{2}\right)-v_3(2u)v_1\left(\alpha\frac{l}{2}\right)}{v_1^2(2u)+v_3^2(2u)}\right]$$

$$\xrightarrow[u=1]{\alpha=2/l}=\frac{pl^3}{EI}\left(\frac{0.609}{48}+\frac{0.111}{8}\cdot\frac{0.9115\times0.6635-4.8301\times1.9335}{1.9115^2+4.9301^2}\right)$$

$$=0.0086Pl^3/EI$$

2.12 题

1）先计算剖面参数：

$$W = bh^2 \big/ 6$$
$$= 2\times 10^2 \big/ 6 = \frac{100}{3}\left(cm^3\right)$$
$$W_p = \sum_i A_i y_i$$
$$= 2\left(\frac{A}{2}\cdot\frac{h}{4}\right) = bh^2 \big/ 4 = 50\left(cm^3\right)$$
$$(\text{形状系数})\, f = W_p / W$$
$$= \frac{bh^2 \big/ 4}{bh^2 \big/ 6} = \frac{3}{2}$$

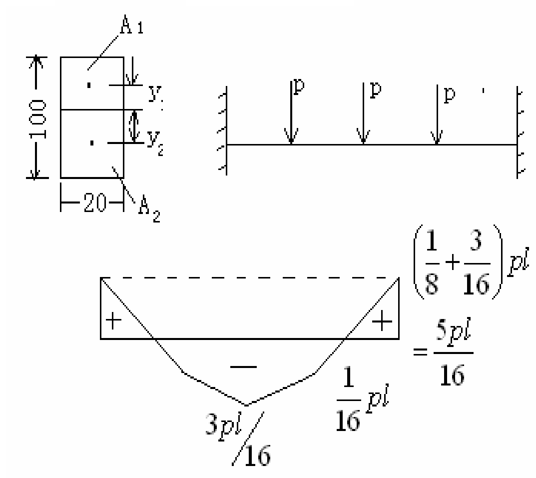


图２．８ａ

2）求弹性阶段最大承载能力$P_{\max}$(如图2.8a)

$$令M_{\max} = W\sigma_y = \frac{100}{3}\times 2400 = 8\times 10^4\, kg \big/ cm^2$$

$$即\frac{5}{16}P_{\max} l = W\sigma_y \quad 解出P_{\max} = \frac{16}{5}\frac{W\sigma_y}{l}\frac{16\times 8\times 10^4}{5\times 500} = 512\left(kg\right)$$

3)求P_u(极限载荷)

(用机动法)此结构
达到极限状态时将
出现三个塑性铰，
其上作用有塑性力
矩$M_p = W_p\sigma_y$如图由虚功原理：

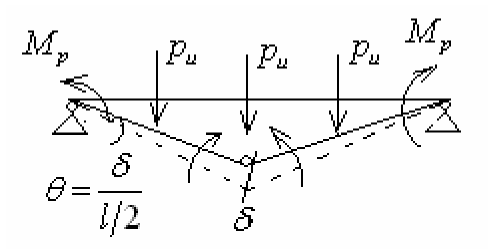


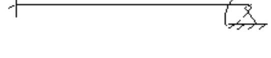
图２．８ｂ

$$P_u\cdot\delta + 2\left(P_u\cdot\delta \big/ 2\right) = 4M_p\left(\frac{\delta}{l \big/ 2}\right)$$

$$\therefore P_u = 4M_p \big/ l = 4W_p\sigma_y \big/ l = 4\times 2400\times 50 \big/ 500 = 960\left(kg\right)$$

2.13 补充题

剪切对弯曲影响补充题，求图示结构剪切影响下的 v(x)



解：可直接利用

$$v(x) = v_0 + \theta_0 x + \frac{M_0 x^2}{2EI} + \frac{N_0}{6EI}\left(x^3 - \frac{6EI}{GA_s}x\right)$$

则边界条件：$v_0=0 \quad \theta_0=0 \quad v(l)=0 \quad EIv''(l)=m$

得 $N_0=\dfrac{3ml}{2l^2+6EI/GA_s} \quad M_0=m-\dfrac{3ml^2}{2l^2+6EI/GA_s}$

$$\therefore v(x)=\frac{m}{EI}\left[\frac{-3\alpha lx}{2l^2+6\alpha}+\frac{6\alpha-l^2}{2l^2+6\alpha}x^2+\frac{lx^3}{2(2l^2+6\alpha)}\right] \qquad \alpha=\frac{EI}{GA_s}$$

2.14. 补充题

试用静力法及破坏机构法求右图示机构的极限载荷 p，已知梁的极限弯矩为 M_p

（20 分）（1983 年华中研究生入学试题）

解： 1）用静力法：（如图２．９）

由对称性知首先固端和中间支座达到塑性铰，再加力 $p\to p_u$，当 p 作用点处也形成塑性铰时结构达到极限状态。即：

$$p_u l/4-M_p=M_p \qquad \therefore p_u=8M_p/l$$

2）用机动法： $2p\delta=8M_p\cdot 2\delta/l \qquad \therefore p_u=8M_p/l$

2.15.补充题

求右图所示结构的极限载荷其中 $\alpha=l/3EI$， $p=ql$（1985 年哈船工研究生入学试题）

解：由对称性只需考虑一半，用机动法。当此连续梁中任意一个跨度的两端及中间发生三个塑性铰时，梁将达到极限状态。考虑 a) 、b)两种可能：

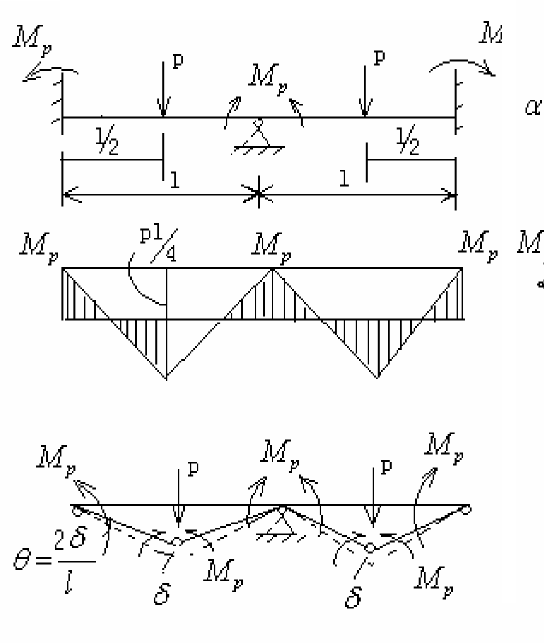
$$对a) \quad 2\int_0^{\frac{l}{2}} q_u\cdot\left(\frac{2\delta}{l}\right)xdx-4M_p\frac{2\delta}{l}=0$$

$$解得 \quad q_u=16M_p/l^2$$

$$对b) \quad p_u\cdot\delta-4M_p\frac{2\delta}{l}=0$$

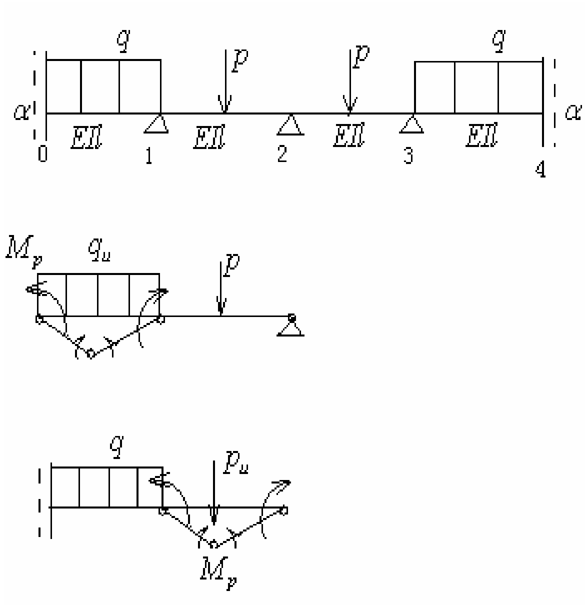
$$\therefore q_u=16M_p/l^2$$

（如图２．１０）取小者为极限载荷为 $q_u=8M_p/l^2$ 即承受集中载荷 p 的跨度是破坏。



M_p
p
M_p
p
M
1/2
1/2
1
1
M_p
$pl/4$
M_p
M_p
M_p
p
M_p
p
M_p
$\theta=\frac{2\delta}{l}$
δ
M_p
δ
M_p

图２．９



q
p
p
q
α
α
EIl
EIl
EIl
EIl
0
1
2
3
4
M_p
q_u
p
q
p_u
M_p

图２．１０

第 3 章　杆件的扭转理论

3.1 题

a)　由狭长矩形组合断面扭转惯性矩公式：

$$J=\frac{1}{3}\sum_i h_i t_i^3=\frac{1}{3}\left[650\times10^3+200\times8^3+80\times8^3\right]=26.4cm^4$$

b)　$J==\frac{1}{3}\left[70\times1.2^3+35\times1^3+15\times1.2^3\right]=60.6cm^4$

c)　由环流方程

$$\varphi'=\oint\tau\,ds\Big/2AG\xrightarrow[f=M_t/2A]{Bredt公式}=\frac{M_t}{4A^2G}\oint\frac{ds}{t}\xrightarrow{材力}=M_t\Big/GJ_0\ \therefore J_0=4A^2\Big/\oint\frac{ds}{t}$$

$$本题A=40\times41.6+\pi\left(20+0.8\right)^2=3023.2\left(cm^2\right)$$

$$\oint\frac{ds}{t}=\frac{1}{1.6}\left(2\times40+41.6\pi\right)=131.68$$

$$\therefore J_0=4\times\left(3023.2\right)^2\Big/131.68=2.775\times10^5\ cm^4$$

3.2 题

对于 a)示闭室其扭转惯性矩为 $J_0=\dfrac{4A^2}{\oint\frac{ds}{t}}=\dfrac{4\left(a-t\right)^4}{\frac{4}{t}\left(a-t\right)}=t\left(a-t\right)^3$

对于 b)开口断面有 $J=\dfrac{1}{3}\sum h_i t_i^3=\dfrac{t^3}{3}\left[4\left(a-t\right)\right]$

$\therefore$两者扭转之比为

$$\varphi_b'\Big/\varphi_a'=\frac{M_t/GJ}{M_t/GJ_0}=J_0/J=\frac{3}{4}\left(\frac{a-t}{t}\right)^2=271(倍)$$

本题易将$\oint\frac{ds}{t}$的积分路径取为截面外缘使答案为300倍，误差为10%，可用但概念不对。若采用s为外缘的话，J大，τ小偏于危险。

3.3 题

$$M_t=\sum_{n=1}^{8}p\frac{b}{2}=8\times\frac{b}{2}\times p=4pb$$

$$A=\frac{8}{2}\left[\left(b-t\right)\sin\frac{\pi}{8}\right]\left[\frac{1}{2}\left(b-t\right)\cos\frac{\pi}{8}\right]=\left(b-t\right)^2\sin\frac{\pi}{4}$$

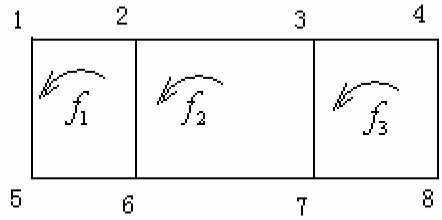
$$\therefore f=M_t\Big/2A=\frac{4bp}{2\left(b-t\right)^2\sin\frac{\pi}{4}}=\frac{2\times100\times30\sqrt{2}}{(300-0.2)^2}=9.555kg/cm$$

$$\therefore \varphi = \frac{l}{2AG}\oint \frac{f}{t}ds = \frac{lf}{2AGt}\left[8(b-t)\sin\frac{\pi}{8}\right] = \frac{100\times 9.56\times 8(b-t)\sin\frac{\pi}{8}}{2(b-t)^2\cdot 2\sin\frac{\pi}{8}c\cos\frac{\pi}{8}}$$

$$= \frac{100\times 9.56\times 8}{4\times 29.8\times c\os\frac{\pi}{8}\cdot 8\times 10^5\times 0.2} = 4\times 10^{-4}(\text{弪})$$

3.4 题

.将剪流对内部任一点取矩



$$\begin{pmatrix} A_1 = a^2/2 = A_3 \\ A_2 = a^2 = 2A_1 \end{pmatrix}$$

$$\int_{2156} f_1 rds + \int_{62}(f_1 - f_2)rds + \int_{32} f_2 rds$$

$$+\int_{67} f_2 rds + \int_{73}(f_2 - f_3)rds + \int_{7843} f_3 rds$$

$$= \oint_{21562} f_1 rds + f_2\oint_{32673} rds + f_3\oint_{78437} rds$$

$$= f_1\oint_{I} rds + f_2\oint_{II} rds + f_3\oint_{III} rds$$

$$= 2A_1 f_1 + 2A_2 f_2 + 2A_3 f_3 = M_t ....(1)$$

由于 I 区与 II 区，II 区与 III 区扭率相等可得两补充方程

$$\frac{1}{2GA_1}\left[\oint\frac{f_1}{t}ds + \oint\frac{-f_2}{t}ds\right] = \frac{1}{2GA_2}\left(\oint_{II}\frac{f_2}{t}ds - \int_{26}\frac{f_1}{t}ds - \int_{73}\frac{f_3}{t}ds\right)$$

$$= \frac{1}{2GA_3}\left(\oint_{III}\frac{f_3}{t}ds + \int_{37}\frac{-f_2}{t}ds\right)$$

即：$\dfrac{3f_1 - f_2}{A_1} = \dfrac{2f_2 + f_1 - f_3}{A_2} = \dfrac{f_3 + f_2}{A_3}$.....(2)

(1)(2)联立（注意到$A_1 = A_3$，$2A_1 = A_2 = a^2$）

$$\begin{cases} 2A_1(f_1 + 2f_2 + f_3) = M_t \\ 3f_1 - f_2 = 3f_3 - f_2 \\ 3f_1 - f_2 = \frac{1}{2}(-f_1 + 4f_2 - f_3) \end{cases} \quad \text{解得} \quad \begin{cases} f_1 = f_3 = \dfrac{3M_t}{14a^2} \\ f_2 = 2M_t/7a^2 \end{cases}$$

$$\therefore \varphi' = \varphi_1' = \frac{1}{2GA_1}\left(\oint\frac{f_1}{t}ds - \int_{62}\frac{f_2}{t}ds\right) = \frac{a}{2G\frac{a^2}{2}t}\left(\frac{9M_t}{14a^2} - \frac{2M_t}{7a^2}\right) = \frac{5M_t}{14a^3 tG}$$

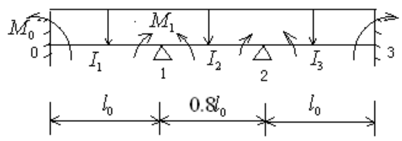
$$\therefore \varphi' = {M_t}\big/{J_0 G} \quad \text{知} \quad J_0 = \frac{14}{5}a^3 t$$

第 4 章　力法

4.1 题

令$l = l_0 = 2.75cm \quad I = I_0 \quad I_2 = 26I_0$

由对称性考虑一半



$q = \left(1 + \dfrac{2.5}{2}\right) \times 0.8 \times 1.025 = 1.845$吨/米

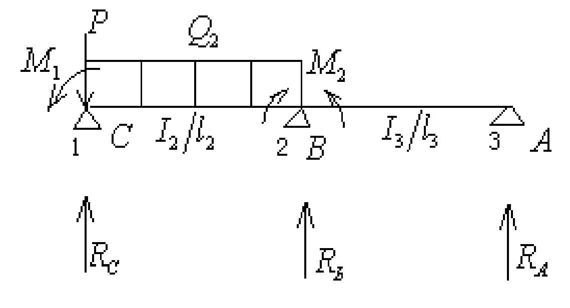
对0,1节点列力法方程

$$\begin{cases} -\dfrac{M_0 l_0}{3EI_0} - \dfrac{M_1 l_0}{6EI_0} - \dfrac{ql_0^3}{24EI_0} = 0 \\ \dfrac{M_0 l_0}{6EI_0} + \dfrac{M_1 l_0}{3EI_0} - \dfrac{ql_0^3}{24EI_0} = -\dfrac{M_1(0.8)l_0}{3E(26I_0)} - \dfrac{M_2(0.8)l_0}{6E(26I_0)} + \dfrac{q_0(0.8l_0)^3}{24E(26I_0)} \end{cases}$$

即：$\begin{cases} M_0 + M_1/2 = ql^2/8 \\ M_0 + 2.09M_1 = 0.2549ql^2 \end{cases}$

$\therefore \begin{cases} M_1 = 0.0817ql^2 = 1.139(t \cdot m) \\ M_0 = 0.0842ql^2 = 1.175(t \cdot m) \end{cases}$

4.2. 题



将第一跨载荷向c支座简化

$\mathrm{M}_1 = Q_1 l_1/2$，　$p = Q_1$

由2节点转轴连续条件：

$$\frac{(Q_1 l_1/2)l_2}{6EI_2} + \frac{M_2 l_2}{3EI_2} - \frac{Q_2 l_2^2}{24EI_2} = \frac{-M_2 l_3}{3EI_3}$$

解得　$M_2 = \dfrac{Q_1 l_1}{8}\left(\dfrac{Q_2 l_2}{Q_1 l_1} - 2\right) \Big/ \left(1 + \dfrac{I_2 l_3}{I_3 l_2}\right)$

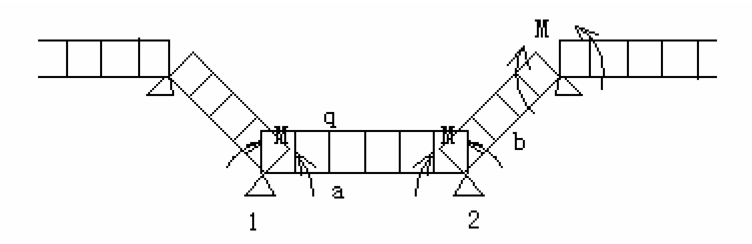
若不计各跨载荷与尺度的区别则简化为$M_2 = {-Ql}/{8\times 2} = {-Ql}/{16}$

$$\begin{cases} R_A = -M_2/l = Q/16 \\ R_B = \left(\dfrac{Q}{2} + \dfrac{M_2 - M_1}{l}\right) + \dfrac{M_2}{l} = -Q/8 \end{cases}$$

4.3 题

由于折曲连续梁足够长且多跨在 a, b 周期重复。可知各支座断面弯矩且为 M 对 2 节点列角变形连续方程

$$\frac{Ma}{3EI} + \frac{Ma}{6EI} - \frac{qa^3}{24EI} = -\frac{Mb}{3EI} - \frac{Mb}{6EI} + \frac{qb^3}{24EI}$$ 解得



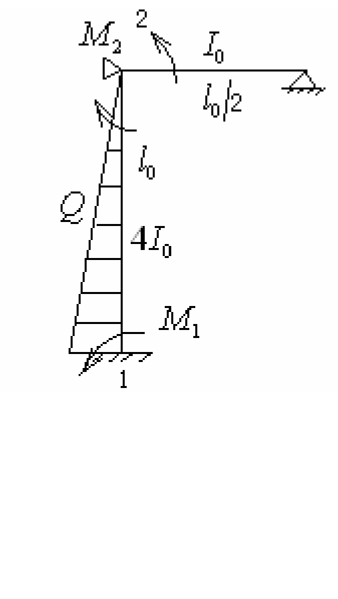
$$M = \frac{q}{12}\left(\frac{a^3 + b^3}{a+b}\right) = \frac{q}{12}\left(a^2 - ab + b^2\right) = \frac{qb^2}{12}\left(1 - \frac{a}{b} + \left(\frac{a}{b}\right)^2\right)$$

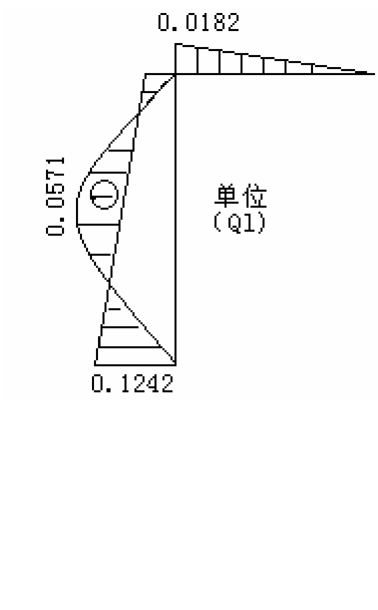
4． 4 题

图4.4°，对2，1节点角连续方程：

$$\begin{cases} \dfrac{M_1(l_0)}{6E(4I_0)} + \dfrac{M_2(l_0)}{3E(4I_0)} - \dfrac{7Q(l_0)^2}{180E(4I_0)} = \dfrac{M_1(l_0/2)}{3EI_0} \\ -\dfrac{M_1(l_0)}{3E(4I_0)} - \dfrac{M_2(l_0)}{6E(4I_0)} + \dfrac{8Q(l_0)^2}{180E(4I_0)} = 0 \end{cases}$$

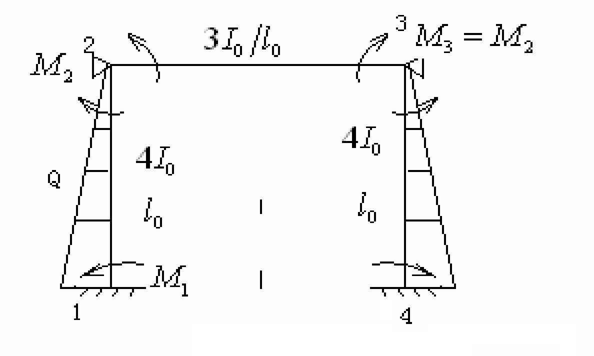
解得：$\begin{cases} M_1 = \dfrac{41}{330}Ql = 0.1242Ql \\ M = Ql/55 = 0.0182Ql \end{cases}$

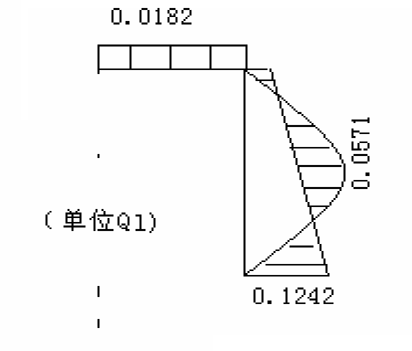




4.5°图

令$I_{12}=I_{34}=4I_0$， $I_{23}=3I_0$　$l_{12}=l_{23}=l_{34}=l_0$,由对称考虑一半





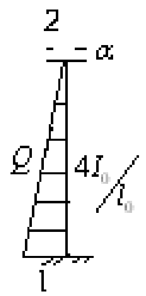
$$\begin{cases} -\dfrac{M_1(l_0)}{3E(4I_0)}-\dfrac{M_2(l_0)}{6E(4I_0)}-\dfrac{2Q(l_0)^2}{45E(4I_0)}=0 \\ \dfrac{M_1(l_0)}{6E(4I_0)}+\dfrac{M_2(l_0)}{3E(4I_0)}-\dfrac{7Q(l_0)^2}{180E(4I_0)}=-\dfrac{M_2l_0}{3E(3I_0)}-\dfrac{M_2l_0}{6E(3I_0)} \end{cases}$$

解出：$\begin{cases} M_1=\dfrac{41}{330}Ql=0.1242Ql \\ M_2=Ql/55=0.0182Ql \end{cases}$

4.5 题

对图4.4°刚架

$$\alpha_2=\frac{1\cdot(l_0/2)}{3EI_0}=\frac{l_0}{6EI_0}$$



对图4.5°所示刚架考虑2,3 杆，由对称性

$$\theta_2=\frac{M_2l_0}{3E(3I_0)}+\frac{M_2l_0}{6E(3I_0)}=\frac{M_2l_0}{6EI_0}$$

$\therefore \alpha_2=l_0/6EI_0$ ∴均可按右图示单跨梁计算。

由附录表A-6（5） $\overline{\alpha}_1=0$　$\overline{\alpha}_2=\dfrac{l_0}{6EI_0}\cdot\dfrac{E(4I_0)}{l_0}=\dfrac{2}{3}$

$$K=\left(0+\frac{1}{3}\right)\left(\frac{2}{3}+\frac{1}{3}\right)-\frac{1}{36}=\frac{11}{36}$$

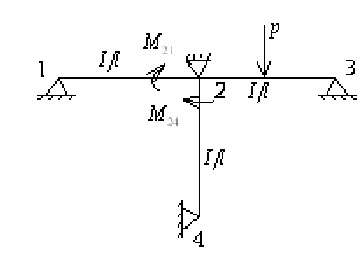
$$\begin{cases} M_1=\dfrac{2Ql_0}{45}\left(\dfrac{1}{11/36}\right)\left(\dfrac{2}{3}+\dfrac{9}{16}\right)=\dfrac{41Ql_0}{330}=0.1242Ql_0 \\ M_2=\dfrac{7Ql_0}{180}\left(\dfrac{1}{11/36}\right)\left(0+\dfrac{1}{7}\right)=\dfrac{Ql_0}{55}=0.0182Ql_0 \end{cases}$$

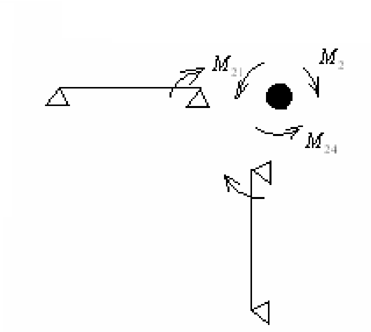
4.6 题

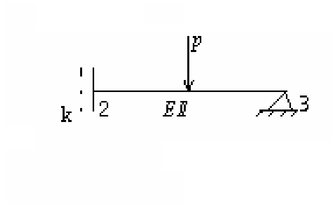
$\because \theta_2$为刚节点，转角唯一（不考虑23杆）

$\therefore \dfrac{M_{21}l}{3EI}=\theta_2=\dfrac{M_{24}l}{3EI}$

$\therefore M_{21}=M_{24}\xrightarrow{\text{2节点平衡}}=M_2/2$







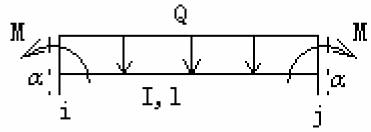
$\therefore \theta_2=\dfrac{(M_2/2)l}{3EI}=\dfrac{M_2 l}{6EI},\therefore \alpha_2=\dfrac{\theta_2}{M_2}=\dfrac{l}{6EI}\qquad K=\dfrac{6EI}{l}$

若21杆单独作用，$K_{21}=\dfrac{1}{\alpha_{21}}=\dfrac{3EI}{l}$,若24杆单独作用，$K_{24}=\dfrac{1}{\alpha_{24}}=\dfrac{3EI}{l}$

$\therefore$两杆同时作用，$K=K_{21}+K_{24}=\dfrac{6EI}{l}$

4.7.题

已知：受有对称载荷 Q 的对称弹性固定端单跨梁（EIl），证明：相应固定系数χ与α关系为：$\chi=1\Big/\left(1+\dfrac{2\alpha EI}{l}\right)$



证：梁端转角$\theta_i=\alpha M=-\dfrac{Ml}{3EI}-\dfrac{Ml}{6EI}+\theta(Q)$

$\therefore M=\theta(Q)\Big/\left(\alpha+\dfrac{l}{2EI}\right)$............................(1)

令$\alpha=0$则相应$M=\bar{M}$(固端弯矩)

即$\bar{M}=\theta(Q)\Big/\dfrac{l}{2EI}$......................................(2)

$(1)/(2)$得$\chi=\dfrac{M}{\bar{M}}=\dfrac{l/2EI}{\alpha+l/2EI}=\dfrac{1}{1+\dfrac{2\alpha EI}{l}}\xrightarrow{\bar{\alpha}=\frac{\alpha}{EI/l}}\dfrac{1}{1+2\bar{\alpha}}$或：$\bar{\alpha}=\dfrac{1}{2}\left(\dfrac{1}{\chi}-1\right)$

讨论：

1）只要载荷与支撑对称，上述结论总成立

2）当载荷与支撑不对称时，重复上述推导可得

$$\chi_i=\frac{\left(2\lambda_{ij}+1\right)\chi_{ij}}{2\lambda_{ij}\chi_{ij}\left(1+3\bar{\alpha}_i\right)+1}\quad or\quad \bar{\alpha}_i=\frac{\frac{\lambda_{ij}}{6}\left(1-\chi_j\right)+\frac{1}{3}}{\chi_i}-\frac{1}{3}$$

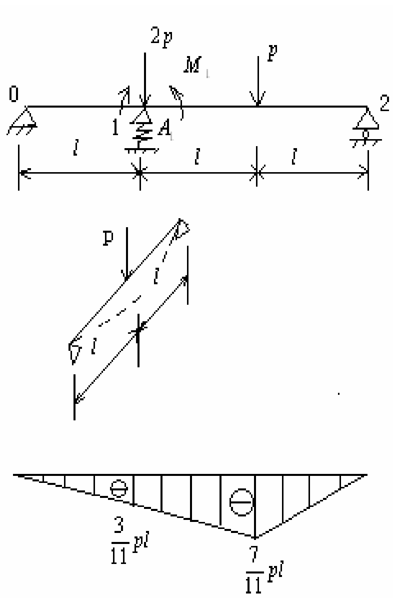
式中$\lambda_{ij}=\bar{M}_i/\bar{M}_j$——外荷不对称系数

$\chi_{ij}=\chi_i/\chi_j$——支撑不对称系数

仅当$\lambda_{ij}=\chi_{ij}=1$即外荷与支撑都对称时有$\chi_i=\dfrac{1}{1+2\bar{\alpha}_i}$

否则会出现同一个固定程度为χ_i的梁端会由载荷不对称或支撑不对称而影响该端的柔度α_i，这与α_i对梁端的约束一定时为唯一的前提矛盾，所以适合$\alpha_i=\theta M_i$定义的$\alpha_i\sim\chi_i$普遍关系式是不存在的。

4.8 题



$A_1=(2l)^3/48EI=l^3/6EI$

列出1节点的角变形连续方程：

$$\begin{cases}\dfrac{M_1 l}{3EI}+\dfrac{v_1}{l}=-\dfrac{M_1(2l)}{3EI}-\dfrac{v_1}{2l}+\dfrac{p(2l)^2}{16EI}\\ v_1=A_1R_1=A_1\left[\left(\dfrac{M_1}{l}+2p\right)+\left(\dfrac{M_1}{2l}+\dfrac{p}{2}\right)\right]\end{cases}$$

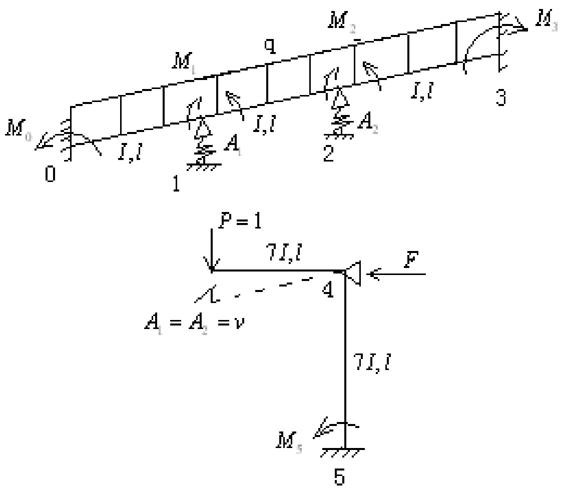
联立解出

$$M_1=-\frac{3}{11}pl,\quad v_1=\frac{23}{36}\frac{pl^3}{EI}$$

画弯矩图见右图

4.9 题

1）如图所示刚架提供的



支撑柔度为$A_1=A_2=V\big|_{p=1}$

而由5节点$\theta_5=0$得

$$-\frac{M_5 l}{3E(7I)}+\frac{(pl)l}{6E(7I)}=0$$

$\therefore M_5=pl/2,$

$$F=\frac{-(pl)-(pl/2)}{l}=-\frac{3p}{2}$$

由卡瓦定理：

$$A=V_1\Big|_{p=1}=\int\frac{M}{EI}\frac{\partial M}{\partial P}ds\Big|_{p=1}$$

$$=\frac{1}{E(7I)}\left[\int_0^l(ps_1)s_1ds_1+\int_0^l\left(pl-\frac{3p}{2}s_2\right)\left(l-\frac{3}{2}s_2\right)ds_2\right]\Big|_{p=1}$$

$$=\frac{1}{E(7I)}\left[\frac{l^3}{3}+\int_0^l\left(l-\frac{3}{2}s_2\right)^2ds_2\right]=\frac{1}{7EI}\left[\frac{l^3}{3}+\frac{l^3}{4}\right]=\frac{l^3}{12EI}$$

2）由对称性只需对0,1节点列出方程组求解

$$\begin{cases}-\dfrac{M_0l}{3EI}-\dfrac{M_1l}{6EI}+\dfrac{v_1}{l}+\dfrac{ql^3}{24EI}=0\\ \dfrac{M_0l}{6EI}+\dfrac{M_1l}{3EI}+\dfrac{v_1}{l}-\dfrac{ql^3}{24EI}=-\dfrac{M_1l}{3EI}-\dfrac{M_1l}{6EI}+\dfrac{ql^3}{24EI}\\ v_1=A_1R_1=\dfrac{l^3}{12EI}\left[\left(\dfrac{M_1-M_0}{l}+\dfrac{ql}{2}\right)+\dfrac{ql}{2}\right]\end{cases}$$

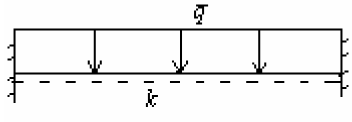
联立解得：$\mathrm{M}_0=11ql^2/36$，$\mathrm{M}_1=-ql^2/36$，$v_1=2v_2=\dfrac{ql^4}{18EI}$

4．10题

a) $\beta=1/384$，$\gamma=1/192$，$Q=qal$

$$\bar{q}=\frac{\beta}{\gamma}\frac{Q}{a}=ql/2$$

$$k=192Ei/al^3$$



(a)

b) $Q=Q_1+Q_2=qal+{qal}/{2}={3qal}/{2}$

$$\therefore Q_1=\frac{2}{3}Q\quad,\quad Q_2=\frac{1}{3}Q$$

$$\because v_{中}=\frac{5Q_1l^3}{384Ei}+\frac{Q_2l^3}{180Ei}\left(\frac{7}{2}+\frac{3}{2^5}-\frac{10}{8}\right)$$

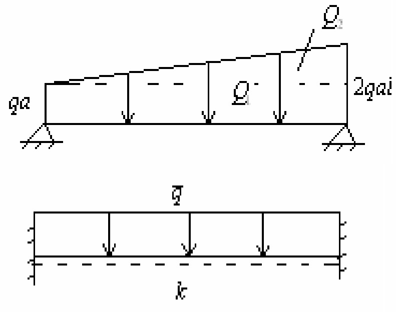
$$=\frac{5Q_1l^3}{384Ei}+\frac{5Q_2l^3}{384Ei}=\frac{Ql^3}{Ei}\left(\frac{5\times{2}/{3}}{384}+\frac{{5}/{3}}{384}\right)$$

$$=\frac{5Ql^3}{384Ei}$$

$$\therefore\beta={5}/{384},\quad\gamma={1}/{48}$$

$$\bar{q}=\frac{\beta}{\gamma}\frac{Q}{a}=\frac{5\times48}{384}\cdot\frac{3qal}{2a}=\frac{15}{16}ql$$

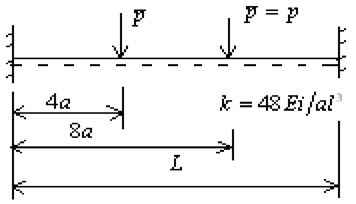
$$k={Ei}/{\gamma al^3}=48Ei/al^3$$



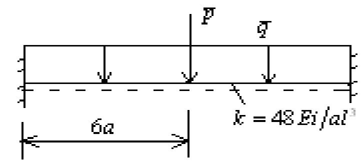
(b)

c) $\beta = 1/48$， $\gamma = 1/48$， $\overline{p} = p$， $Q = p$ $\quad \therefore k = Ei/\gamma a l^3 = 48Ei/al^3$

d)令 $\dfrac{\overline{p}l^3}{48Ei} = \dfrac{pl^2\frac{l}{4}}{6Ei}\left(\dfrac{3}{4}-\dfrac{1}{4^2}\right) \therefore \overline{p} = \dfrac{48p}{6}\cdot\dfrac{1}{4}\cdot\dfrac{1}{4}\left(3-\dfrac{1}{4}\right) = \dfrac{11}{8}p, k = 48Ei/al^3$（同c图）



(c)



(e)

e) $\beta = 5/384$， $\gamma = 1/48$， $Q = qal/2$ $\quad \therefore \overline{q} = \dfrac{\beta}{\gamma}\dfrac{Q}{a} = \dfrac{5\times 48}{384}\cdot\dfrac{qal}{2a} = \dfrac{5}{16}ql$

令 $\dfrac{\overline{p}l^3}{48Ei} = \dfrac{pl^3}{6Ei}\left[\dfrac{1}{3}\times\dfrac{1}{2}\left(1-\dfrac{1}{4}-\dfrac{1}{9}\right)\right] \quad \therefore \overline{p} = \dfrac{48}{6}p\left[\dfrac{1}{6}\times\dfrac{23}{36}\right] = \dfrac{23}{27}p$

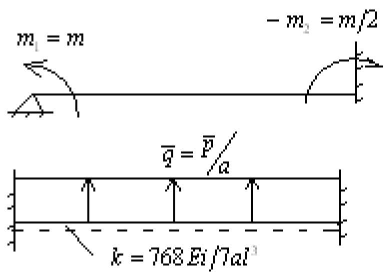
f) 令 $\dfrac{7\overline{p}l^3}{768Ei} = -\dfrac{l^2}{6Ei}\cdot\dfrac{1}{2}\left(1-\dfrac{1}{2}\right)\left[m\left(2-\dfrac{1}{2}\right)+\left(-\dfrac{m}{2}\right)\left(1+\dfrac{1}{2}\right)\right]$

$\therefore \overline{p} = -\dfrac{768}{7\times 12}\times\dfrac{1}{2}\left(\dfrac{3}{2}-\dfrac{1}{2}\times\dfrac{3}{2}\right)\dfrac{m}{l} = -\dfrac{24}{7}\dfrac{m}{l} \qquad k = 768Ei/7al^3$

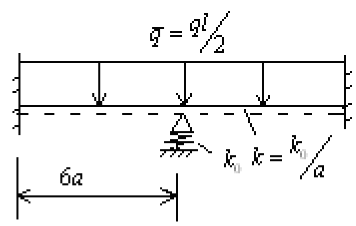
g) $\overline{p}$同 a)即 $\overline{p} = \dfrac{\theta}{\gamma}Q = \dfrac{qal}{2} \quad \therefore \overline{q} = \overline{p}/a = \dfrac{ql}{2}$

$$k = \frac{1}{A} = \begin{cases} 192Ei/l^3 = k_0 & (x \neq 6a) \\ 192E(2i)/l^3 = 2\times\dfrac{192Ei}{l^3} = 2k_0 & (x = 6a) \end{cases}$$

$\therefore k = k_0/a = 192Ei/al^3$



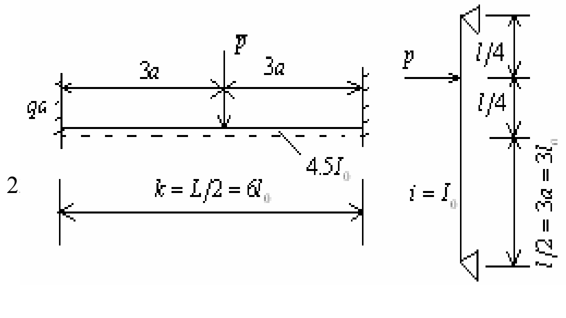
(f)



(g)

4.11 题

∵支柱处$v = \theta = 0$,可简化为刚性固定约束∴仅考虑右半边板架



$$\gamma = \frac{1}{48}$$

$$\beta = \frac{1}{6}\left[\frac{1}{4}\cdot\frac{1}{2}\left(1-\frac{1}{16}-\frac{1}{4}\right)\right] = \frac{11}{48\times16}$$

$$\overline{p} = \frac{\beta}{\gamma}p = \frac{11}{16}p$$

$$k = 48Ei/al^3 = 48EI_0/l_0(6l_0)^3 = \frac{2}{9}\frac{EI_0}{l^4{}_0}$$

$$u = \frac{(6l_0)}{2}\sqrt[4]{\frac{\left(\frac{2}{9}\frac{EI_0}{l^4{}_0}\right)}{(4E9I_0/2)}} = 1$$

$$\therefore \bar{M} = \frac{\overline{p}(6l_0)}{8}\lambda_1(1) = \frac{\frac{11}{16}p\cdot(6l_0)}{8}\times0.874 = 0.4507pl_0$$

$$\bar{N} = \pm\frac{\overline{p}}{2}\varphi_1(1) = \pm\frac{\overline{p}}{2}\times0.852 = \frac{11}{32}P\times0.852 = \pm0.2929p$$

$$\nu_{\max} = \nu\left(\frac{l}{4}\right) = \nu(3l_0)$$

$$= \frac{\overline{p}(6l_0)^3}{192E\left(\frac{9}{2}I_0\right)}\eta_1(1) = \frac{\frac{11}{16}\times6^3\times0.889}{96\times9}\cdot\frac{pl^3{}_0}{EI_0} = 0.1528\frac{pl^3{}_0}{EI_0}$$

4.12 题

设$a = l_0 = 1m \quad i = I_0 = 5.833\times10^5\,cm^4 \quad L_1 = l = 10l_0 \quad b = 2.5l_0$

$I = 1.857I_0, \quad Q = q_0al, \quad q_0 = 1kg/cm^2 \quad E = 2\times10^6\,kg/cm^2$

'求：中纵桁跨中及端部弯曲应力及$\nu_{中}$

解：因主向梁两端简支受均布载荷Q故其形状可设为$\sin\frac{\pi y}{1}$

$$c_1 = c_3 = \sin\frac{\pi y_1}{1} = \sin\frac{\pi}{4} = 0.707 \quad c_2 = \sin\frac{\pi}{2} = 1$$

$$\gamma_{11}=\frac{1}{6}\left[\frac{1}{4}\left(3\times\frac{1}{4}-\frac{1}{4}\right)\right]=0.02083 \quad (\text{按对称跨中求})$$

$$\gamma_{12}=\frac{1}{6}\left[\frac{1}{2}\times\frac{1}{4}\left(1-\frac{1}{16}-\frac{1}{4}\right)\right]=0.01432$$

$$\gamma^*{}_1=\sum_{i=1}^{2}\gamma_{1i}c_iI_i/c_1I_1=\frac{1}{0.707}\left[0.02083\times0.707+0.01432\times1\right]=0.0411$$

$$\gamma^*{}_2=\gamma^*{}_1I_1/I_2=\gamma^*{}_1=0.0411,\quad \beta_2=4/384=0.01302$$

$$k_2=Ei/\gamma^*{}_2al^3=2\times10^6\times5.833\times10^5/0.0411\times10^2\times\left(10^3\right)^3=283\,kg/cm$$

$$\overline{q}_2=\frac{\beta_2}{\gamma^*{}_2}\frac{Q}{l_0}=\frac{0.01302\times10q_0l_0^2}{0.0411l_0}=3.168q_0l_0$$

$$u=\frac{L}{2}\sqrt[4]{\frac{i}{4al^3I_1\gamma_1^*}}=\frac{10l_0}{2}\sqrt[4]{I_0/4l_0\left(10l_0\right)^3 1.857I_0\times0.0411}\approx1.2$$

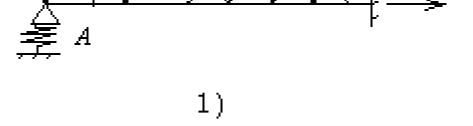
$$\varphi_1(1.2)=0.728,\quad \chi_2(u)=0.813,\quad \chi_1(u)=0.774$$

$$v_{中}=\frac{\overline{q}_2}{k_2}\left(1-\varphi_1(u)\right)=\frac{3.168\times1\times10^2}{283}(1-0.728)=0.304(cm)$$

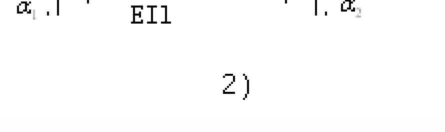
$$\left|\sigma_{端}\right|=\left|\frac{\bar{\mathrm{M}}}{I\Big/\left(\dfrac{h}{2}+t\right)}\right|=\frac{\overline{q}_2L^2}{12}\cdot\frac{\chi_2(1.2)}{\mathrm{I}/51}=\frac{3.168q_0l_0\left(10l_0\right)^2}{12}\times\frac{0.813\times51}{10.833\times10^5}=1010\,kg/cm$$

$$\left|\sigma_{中}\right|=\frac{\overline{q}_2L^2}{24}\cdot\frac{\chi_1(\mathrm{u})}{\mathrm{I}/51}=\frac{3.168q_0l_0\left(10l_0\right)^2}{24}\times\frac{0.774\times51}{10.833\times10^5}=481\,kg/cm$$

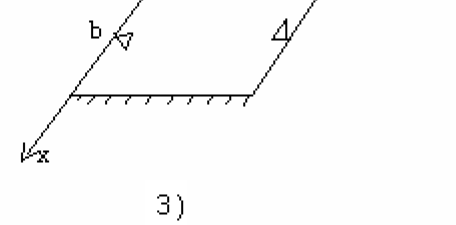
4.13 补充题 写出下列构件的边界条件：（15 分）



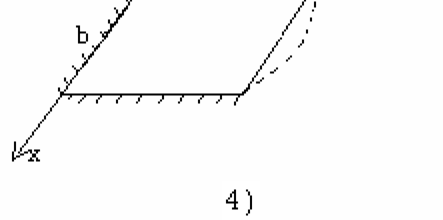
1)



2)



3)



4)

1）

解：$\begin{cases} v(0)=A\left[p_1-EIv'''(0)\right] \\ EIv''(0)=m \end{cases}$ $\begin{cases} v'(l)=0 \\ v(l)=0 \end{cases}$

2）

解：$\begin{cases} v'(0)=\alpha_1\left[EIv''(0)-m_1\right] \\ v(0)=0 \end{cases}$ $\begin{cases} v'(l)=\alpha_2\left[EIv''(l)+m_2\right] \\ v(l)=0 \end{cases}$

3) 设 x=0,b 时两端刚性固定；y=0,a 时两端自由支持

解：x=0,b时$\begin{cases} \dfrac{\partial w}{\partial x}=0 \\ w=0 \end{cases}$ y=0,a时$\begin{cases} \dfrac{\partial^2 w}{\partial y^2}=0 \\ w=0 \end{cases}$

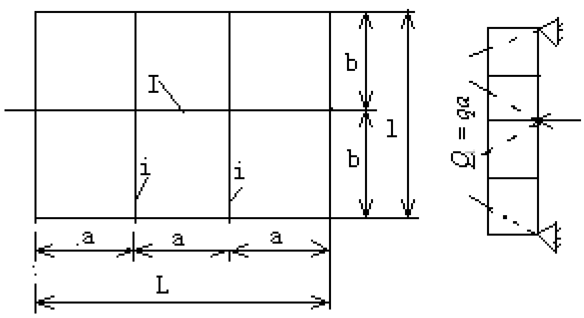
4) 已知：x=0,b 为刚性固定边;y=0 边也为刚性固定边：y=a 为完全自由边

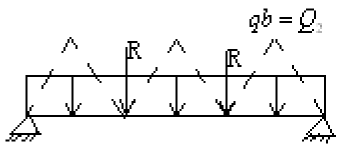
解：x=0,b时$\begin{cases} \dfrac{\partial w}{\partial x}=0 \\ w=0 \end{cases}$

y=0时$w=\dfrac{\partial w}{\partial y}=0$

$y=a$时$\begin{cases} \dfrac{\partial^2 w}{\partial y^2}+\mu\dfrac{\partial^2 w}{\partial x^2}=0 \\ \dfrac{\partial}{\partial y}\left[\dfrac{\partial^2 w}{\partial y^2}+(2-\mu)\dfrac{\partial^2 w}{\partial x^2}\right]=0 \end{cases}$

4.14 题. 图示简单板架设受有均布载荷 q 主向梁与交叉构件两端简支在刚性支座上，试分析两向梁的尺寸应保持何种关系，才能确保交叉构件对主向梁有支持作用？





解：∵少节点板架两向梁实际承受载荷如图，为简单起见都取为均布载荷。由

对称性：$R_1 = R_2 = R$ 由节点挠度相等：

$$\left.\begin{aligned} w_{主} &= \frac{5}{384}\frac{Q_1 l^3}{Ei} + \frac{1}{48}\frac{Rl^3}{Ei} \\ w_{交} &= \frac{11}{972}\frac{Q_2 L^3}{EI} + \frac{5}{162}\frac{RL^3}{EI} \end{aligned}\right\}\text{使之相等令}$$

$$Q_1 = qal = \frac{1}{3}qlL \quad Q_2 = qbl = \frac{1}{2}qlL$$

解出节点反力$R = qlL\left(\frac{5}{1152}\alpha - \frac{11}{1944}\right)\Big/\left[\frac{\alpha}{48} + \frac{5}{162}\right]$……………………………(1)

式中$\alpha = \frac{l^3 I}{L^3 i}$——交叉构件与主向梁的相对刚度，且$\frac{dR}{d\alpha} > 0$

由(1)节点反力将随α的增加(即交叉构件刚性的增加）而增加。

当$\alpha \to \infty$时$R = R_{max} = \frac{5}{1152} \times 48qlL = \frac{5}{24}qlL$

这时交叉构件对主向梁的作用相当于一个刚性支座

当$\frac{5}{1152}\alpha < \frac{11}{1944}$时即$\frac{I}{L^3} < 1.3\frac{i}{l^3}$时$R < 0$表示交叉构件的存在不仅不支持主向梁，反而加重其负担，使主向梁在承受外载荷以外还要受到向下的节点反作用力这是很不利的。

∴只有当$\frac{I}{L^3} > 1.3\frac{i}{l^3}$时，主向梁才受到交叉构件的支持。

第 5 章 位移法

5.1 题

图 4.4[0] $\overline{M}_{12}=-Ql_0/10$，$\overline{M}_{21}=Ql_0/15$，$\overline{M}_{32}=\overline{M}_{23}=0$

$$M_{12}^{'}=\frac{2E(4I_0)}{l_0}\theta_2\text{，}\quad M_{21}^{'}=\frac{4E(4I_0)}{l_0}\theta_2$$

$$M_{23}^{'}=\frac{2EI_0}{l_0/2}\theta_3+\frac{4EI_0}{l_0/2}\theta_2$$

$$M_{32}^{'}=\frac{4EI_0}{l_0/2}\theta_3+\frac{2EI_0}{l_0/2}\theta_2$$

对于节点 2，列平衡方程

$$\begin{cases} M_{32}=0 \\ M_{23}+M_{21}=0 \end{cases}\qquad \text{即：}\quad \begin{cases} M^{'}{}_{32}+\overline{M}_{32}=0 \\ M^{'}{}_{23}+M^{'}{}_{21}+\overline{M}_{23}+\overline{M}_{21}=0 \end{cases}$$

代入求解方程组，有

$$\begin{cases} \dfrac{4EI_0}{l_0}\theta_2+\dfrac{8EI_0}{l_0}\theta_3=0 \\ (\dfrac{8EI_0}{l_0}+\dfrac{8EI_0}{l_0})\theta_2+\dfrac{4EI_0}{l_0}\theta_3=-\dfrac{Ql_0}{15} \end{cases}\text{，解得}\begin{cases} \theta_2=-\dfrac{Ql_0{}^2}{22\times15EI_0} \\ \theta_3=\dfrac{Ql_0{}^2}{44\times15EI_0} \end{cases}$$

所以 $M_{12}=M_{12}^{'}+\overline{M_{12}}=\dfrac{8EI_0}{l_0}\left[\dfrac{-Ql_0^2}{22\times15EI_0}\right]-\dfrac{Ql_0}{10}=-\dfrac{41}{330}Ql_0=-0.1242Ql_0$

$$M_{21}=M_{21}^{'}+\overline{M_{21}}=\frac{16EI_0}{l_0}\left[\frac{-Ql_0^2}{22\times15EI_0}\right]+\frac{Ql_0}{15}=\frac{Ql_0}{55}=0.0182Ql_0$$

图 4.5^0。 由对称性知道：$\theta_2=-\theta_3=-\theta$

1）$\overline{M}_{12}=-Ql_0/10$，$\overline{M}_{21}=Ql_0/15$，$\overline{M}_{32}=\overline{M}_{23}=0$

2） $M_{12}^{'}=\dfrac{2E(4I_0)}{l_0}\theta_2$，$M_{21}^{'}=\dfrac{4E(4I_0)}{l_0}\theta_2$

$$M_{23}^{'}=\frac{2E(3I_0)}{l_0}\theta_3+\frac{4E(3I_0)}{l_0}\theta_2=\frac{6EI_0}{l_0}\theta_2$$

3） 对 2 节点列平衡方程 $M_{23}+M_{21}=0$

即$\dfrac{16EI_0}{l_0}\theta_2+\dfrac{Ql_0}{15}+\dfrac{6EI_0}{l_0}\theta_2=0$，解得$\theta_2=-\dfrac{Ql_0^{\,2}}{22\times 15EI_0}$

4）求M_{12},M_{21},M_{23}（其余按对称求得）

$$M_{12}=M_{12}^{'}+\overline{M_{12}}=\frac{8EI_0}{l_0}\left[\frac{-Ql_0^{\,2}}{22\times 15EI_0}\right]-\frac{Ql_0}{10}=-\frac{41}{330}Ql_0=-0.1242Ql_0$$

$$M_{21}=M_{21}^{'}+\overline{M_{21}}=\frac{16EI_0}{l_0}\left[\frac{-Ql_0^{\,2}}{22\times 15EI_0}\right]+\frac{Ql_0}{15}=\frac{Ql_0}{55}=0.0182Ql_0$$

$M_{23}=-M_{21}$，其余$M_{43}=-M_{21}$，$M_{34}=-M_{21}$，$M_{32}=-M_{23}$

5.2 题

由对称性只要考虑一半，如左半边

1）固端力（查附表 A-4）

$\bar{M}_{12}=-Q(2l_0)/10=-\dfrac{1}{5}q_0l_0^{\,2}$，　　$\bar{M}_{21}=Q(2l_0)/15=\dfrac{2}{15}q_0l_0^{\,2}$

$\bar{M}_{25}=\bar{M}_{23}=\bar{M}_{32}=\bar{M}_{34}=0$

2）转角θ_2,θ_3对应弯矩（根据公式 5-5）

$M_{12}^{'}=\dfrac{2E(4I_0)}{2l_0}\theta_2$，$M_{21}^{'}=\dfrac{4E(4I_0)}{2l_0}\theta_2$，

$M_{25}^{'}=\dfrac{4EI_0}{4l_0}\theta_2+\dfrac{2EI_0}{4l_0}\theta_5\underset{\theta_5=-\theta_2}{=}\dfrac{EI_0}{2l_0}\theta_2$

$M_{23}^{'}=\dfrac{4EI_0}{l_0}\theta_2+\dfrac{2EI_0}{l_0}\theta_3$，

$M_{32}^{'}=\dfrac{2EI_0}{l_0}\theta_2+\dfrac{4EI_0}{l_0}\theta_3$

$M_{34}^{'}=\dfrac{4EI_0}{4l_0}\theta_3+\dfrac{2EI_0}{4l_0}\theta_4\underset{\theta_4=-\theta_3}{=}\dfrac{EI_0}{2l_0}\theta_3$

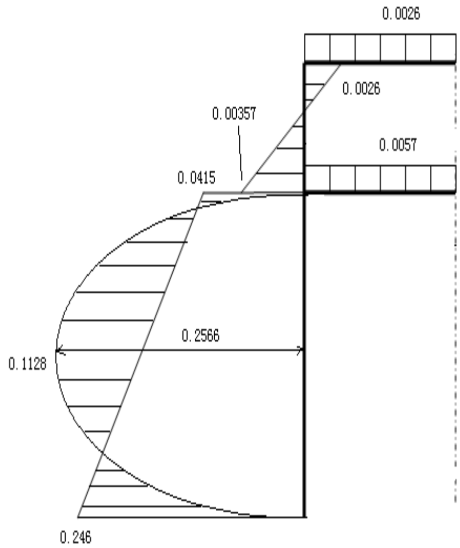


图 5.1 （单位：$q_0l_0^{\,2}$）

3）对于节点 2，3 列出平衡方程

$$\begin{cases} M_{32}+M_{34}=0 \\ M_{21}+M_{25}+M_{23}=0 \end{cases} \quad 即\begin{cases} M'_{32}+M'_{34}=-(\bar{M}_{32}+\bar{M}_{34}) \\ M'_{25}+M'_{23}+M'_{21}=-\left(\bar{M}_{23}+\bar{M}_{21}+\bar{M}_{25}\right) \end{cases}$$

则有$\begin{cases} \dfrac{2EI_0}{l_0}\theta_2+\dfrac{4EI_0}{l_0}\theta_3+\dfrac{EI_0}{2l_0}\theta_3=0 \\ \dfrac{8EI_0}{l_0}\theta_2+\dfrac{EI_0}{2l_0}\theta_2+\dfrac{4EI_0}{l_0}\theta_2+\dfrac{2EI_0}{l_0}\theta_3=-\dfrac{2q_0l_0^2}{15} \end{cases}$，得$\begin{cases} \theta_2=\dfrac{-12}{1045}\dfrac{q_0l_0^3}{EI_0} \\ \theta_3=\dfrac{16}{3\times1045}\dfrac{q_0l_0^3}{EI_0} \end{cases}$

4）

$$M_{12}=M_{12}^{'}+\overline{M_{12}}=\frac{4EI_0}{l_0}\left[\frac{-12}{1045}\frac{q_0l_0^3}{EI_0}\right]+(-\frac{1}{5}q_0l_0^2)=-\frac{257}{1045}q_0l_0^2=-0.246q_0l_0^2$$

$$M_{21}=\frac{8EI_0}{l_0}\left[\frac{-12}{1045}\frac{q_0l_0^3}{EI_0}\right]+\frac{2}{15}q_0l_0^2=\frac{26}{627}q_0l_0^2=0.0415q_0l_0^2$$

$$M_{25}=\frac{EI_0}{2l_0}\left[\frac{-12}{1045}\frac{q_0l_0^3}{EI_0}\right]=-\frac{6}{1045}q_0l_0^2=-0.0057q_0l_0^2$$

$$M_{23}=\frac{4EI_0}{l_0}\left[\frac{-12}{1045}\frac{q_0l_0^3}{EI_0}\right]+\frac{2EI_0}{l_0}\left[\frac{16}{3\times1045}\frac{q_0l_0^3}{EI_0}\right]=-\frac{112}{3135}q_0l_0^2=-0.0357q_0l_0^2$$

$$M_{32}=\frac{2EI_0}{l_0}\left[\frac{-12}{1045}\frac{q_0l_0^3}{EI_0}\right]+\frac{4EI_0}{l_0}\left[\frac{16}{3\times1045}\frac{q_0l_0^3}{EI_0}\right]=-\frac{8}{3135}q_0l_0^2=-0.0026q_0l_0^2$$

其余由对称性可知（各差一负号）：$M_{65}=-M_{12}$，$M_{56}=-M_{21}$，

$M_{52}=-M_{25}$，$M_{54}=-M_{23}$，$M_{45}=-M_{32}$，$M_{43}=-M_{34}=M_{32}$；弯矩图如图5.1

5.3 题

（$M_{14}=M_{25}=0$）$\bar{M}_{12}=-pl/8$，$\bar{M}_{21}=pl/8$，其余固端弯矩都为 0

$M_{41}^{'}=\dfrac{2EI}{l}\theta_1$，$M_{14}^{'}=\dfrac{4EI}{l}\theta_1$，$M_{52}^{'}=\dfrac{2EI}{l}\theta_2$，$M_{25}^{'}=\dfrac{4EI}{l}\theta_2$

$M_{63}^{'}=\dfrac{2EI}{l}\theta_3$，$M_{36}^{'}=\dfrac{4EI}{l}\theta_3$

$M_{12}^{'}=\dfrac{4EI}{l}\theta_1+\dfrac{2EI}{l}\theta_2$，　$M_{21}^{'}=\dfrac{2EI}{l}\theta_1+\dfrac{4EI}{l}\theta_2$

$M_{23}^{'}=\dfrac{4EI}{l}\theta_2+\dfrac{2EI}{l}\theta_3$，　$M_{32}^{'}=\dfrac{2EI}{l}\theta_2+\dfrac{4EI}{l}\theta_3$

由 1、2、3 节点的平衡条件

$\begin{cases} M_{14}+M_{12}=0 \\ M_{21}+M_{25}+M_{23}=0 \\ M_{32}+M_{36}=0 \end{cases}$　即$\begin{cases} M_{14}^{'}+M_{12}^{'}=-\left(\bar{M}_{14}+\bar{M}_{12}\right) \\ M_{25}^{'}+M_{23}^{'}+M_{21}^{'}=-\left(\bar{M}_{23}+\bar{M}_{21}+\bar{M}_{25}\right) \\ M_{32}^{'}+M_{36}^{'}=-\left(\bar{M}_{32}+\bar{M}_{36}\right) \end{cases}$

$$\begin{cases} \frac{4EI}{l}\theta_1 + \frac{4EI}{l}\theta_1 + \frac{2EI}{l}\theta_2 = \frac{pl}{8} \\ \frac{2EI}{l}\theta_1 + \frac{4EI}{l}\theta_2 + \frac{4EI}{l}\theta_2 + \frac{4EI}{l}\theta_2 + \frac{2EI}{l}\theta_3 = -\frac{pl}{8} \\ \frac{2EI}{l}\theta_2 + \frac{4EI}{l}\theta_3 + \frac{4EI}{l}\theta_3 = 0 \end{cases}$$

解得：$\theta_1 = \frac{27}{22\times 64}\frac{pl^2}{EI}$，$\theta_2 = -\frac{5}{22\times 16}\frac{pl^2}{EI}$，$\theta_3 = \frac{5}{22\times 64}\frac{pl^2}{EI}$

$$M_{14} = -M_{12} = \frac{4EI}{l}\left(\frac{27}{22\times 64}\frac{pl^2}{EI}\right) = \frac{27}{352}pl = 0.0767pl$$

$$M_{41} = \frac{2EI}{l}\left(\frac{27}{22\times 64}\frac{pl^2}{EI}\right) = \frac{27}{704}pl = 0.0383pl$$

$$M_{36} = \frac{4EI}{l}\left(\frac{5}{22\times 64}\frac{pl^2}{EI}\right) = \frac{5}{352}pl = 0.0142pl = -M_{32}$$

$$M_{63} = \frac{2EI}{l}\left(\frac{5}{22\times 64}\frac{pl^2}{EI}\right) = \frac{5}{704}pl = 0.007pl$$

$$M_{25} = \frac{4EI}{l}\left(-\frac{5}{22\times 16}\frac{pl^2}{EI}\right) = -\frac{5}{88}pl = -0.0568pl$$

$$M_{23} = \frac{4EI}{l}\left(-\frac{5}{22\times 16}\frac{pl^2}{EI}\right) + \frac{2EI}{l}\left(\frac{5}{22\times 64}\frac{pl^2}{EI}\right) = -\frac{35}{704}pl = -0.0497pl$$

$$M_{21} = -M_{25} - M_{23} = \frac{75}{704}pl = 0.1065pl$$

$$M_{52} = \frac{2EI}{l}\left(-\frac{5}{22\times 16}\frac{pl^2}{EI}\right) = -\frac{5}{176}pl = -0.0284pl$$

弯矩图如图 5.2

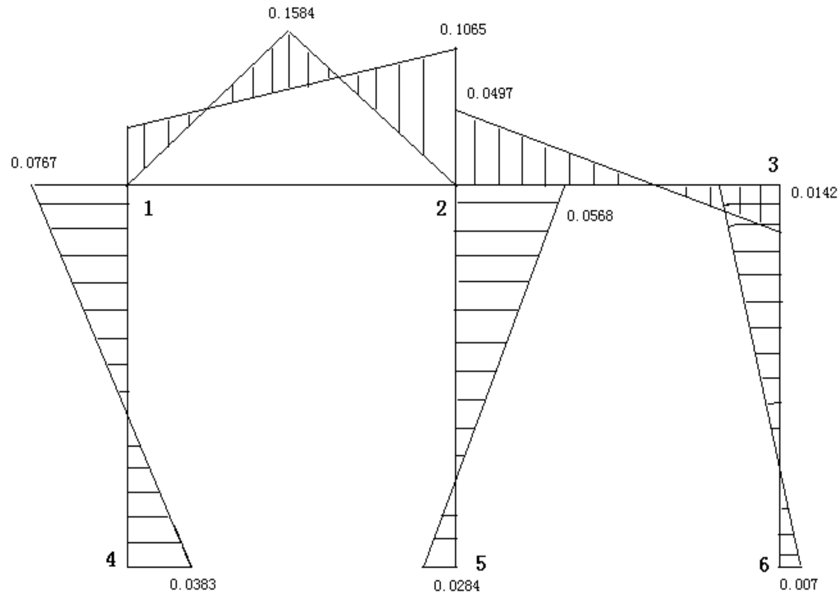


图 5.2（单位：ql）

5.4 题

已知 $l_{12}=l_0=3m$，$l_{23}=2.2l_0=6.6m$，$l_{24}=3l_0=9m$

$$I_0=0.3\times10^4cm^4,\quad I_{12}=2I_0,\quad I_{23}=3I_0,\quad I_{24}=8I_0$$

$$Q_0=\frac{1}{2}q_2l_{12}=\frac{1}{2}q_0l_0,\quad q_4=4q_0,$$

$$Q_{24}=Q^{矩}{}_{24}+Q^{三角}{}_{24}=q_0(3l_0)+\frac{1}{2}(3q_0)3l_0=6Q_0+9Q_0$$

1）求固端弯矩

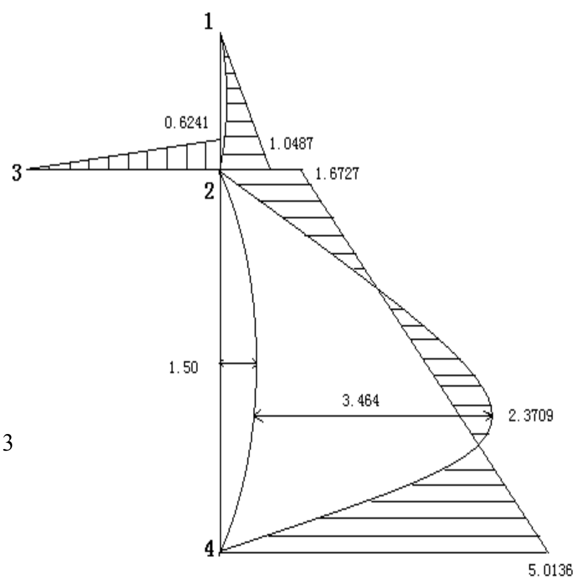
$$\bar{M}_{21}=Q_0l_0/10,\quad \bar{M}_{12}=-Q_0l_0/15,\quad \bar{M}_{32}=\bar{M}_{23}=0$$

$$\bar{M}_{24}=-\frac{(9Q_0)(3l_0)}{15}-\frac{(6Q_0)(3l_0)}{12}=-{}^{33Q_0l_0}\!/_{10}$$

$$\bar{M}_{42}=\frac{(9Q_0)(3l_0)}{10}+\frac{(6Q_0)(3l_0)}{12}={}^{21Q_0l_0}\!/_{5}$$

2） 转角弯矩

$$M'_{12}=\frac{4E(2I_0)}{l_0}\theta_1+\frac{2E(2I_0)}{l_0}\theta_2,$$



$$M'_{21}=\frac{2E(2I_0)}{l_0}\theta_1+\frac{4E\left(2I_0\right)}{l_0}\theta_2$$

$$M'_{23}=\frac{4E(3I_0)}{2\bullet(2l_0)}\theta_2+\frac{2E(3I_0)}{2\bullet(2l_0)}\theta_3\,,$$

$$M'_{32}=\frac{2E(3I_0)}{2\bullet(2l_0)}\theta_2+\frac{4E(3I_0)}{2\bullet(2l_0)}\theta_3$$

$$M'_{24}=\frac{4E(8I_0)}{(3l_0)}\theta_2\,,$$

$$M'_{42}=\frac{2E(8I_0)}{(3l_0)}\theta_2$$

图 5.3（单位：Q_0l_0）

3）对 1、2、3 节点列平衡方程

$$\begin{cases}M_{12}=0\\M_{21}+M_{24}+M_{23}=0\\M_{32}=0\end{cases}\text{即：}\begin{cases}\dfrac{8EI_0}{l_0}\theta_1+\dfrac{4EI_0}{l_0}\theta_2=Q_0l_0/15\\\dfrac{4EI_0}{l_0}\theta_1+\dfrac{796}{33}\dfrac{EI_0}{l_0}\theta_2+\dfrac{30}{11}\dfrac{EI_0}{l_0}\theta_3=-\left(-\dfrac{16}{5}Q_0l_0\right)\\\dfrac{30}{11}\dfrac{EI_0}{l_0}\theta_2+\dfrac{60}{11}\dfrac{EI_0}{l_0}\theta_3=0\end{cases}$$

解得：$\theta_1=-\dfrac{2234}{32880}\dfrac{Q_0l_0^2}{EI_0}=-0.03397\dfrac{q_0l_0^2}{EI_0}$，$\theta_2=\dfrac{209}{1370}\dfrac{Q_0l_0^2}{EI_0}=0.07628\dfrac{q_0l_0^2}{EI_0}$，

$$\theta_3=-\frac{209}{2740}\frac{Q_0l_0^2}{EI_0}=-0.03814\frac{q_0l_0^2}{EI_0}$$

4）求出节点弯矩

$$M_{21}=\left(-\frac{4\times2234}{32880}+\frac{8\times209}{1370}+\frac{1}{10}\right)Q_0l_0=1.0487Q_0l_0$$

$$M_{23}=\left(\frac{12}{1.2}\times\frac{209}{1370}-\frac{6}{2.2}\times\frac{209}{2740}\right)Q_0l_0=0.6241Q_0l_0$$

$$M_{24}=\left(\frac{32}{3}\times\frac{209}{1370}-\frac{33}{10}\right)Q_0l_0=-1.6727Q_0l_0$$

$$M_{24}=\left(\frac{14}{3}\times\frac{209}{1370}+\frac{21}{5}\right)Q_0l_0=5.0136Q_0l_0$$

弯矩图如图 5.3。

5.5 题

由对称性只考虑一半；

节点号	1	2	
杆件号 ij	12	21	23
I_{ij}/I_0	——	4	3
l_{ij}/l_0	——	1	1
k_{ij}	——	4	3
C_{ij}	——	1	（1/2）对称
$C_{ij}k_{ij}$	——	4	3/2
$\sum C_{ij}k_{ij}$	——	11/2	
λ_{ij}	——	8/11	3/11
n_{ij}	——	1/2	——
$\bar{M}_{ij}/Ql_0$	-1/10	1/15	0
m_{ij}/Ql_0 m'_{ij}/Ql_0	-4/165	-8/165	-1/55
M_{ij}/Ql_0	-41/330	1/55	-1/55

所以：

$$M_{12}=-M_{43}=-\frac{41Ql_0}{330}，\quad M_{21}=-M_{34}=\frac{Ql_0}{55}，\quad M_{23}=-M_{32}=-\frac{Ql_0}{55}$$

5.6 题

1.图 5.4⁰：令 $I_{10}=I_0=I_{12},l_{10}=l_0,l_{12}=1.5l_0$

节点号	0	1		2
杆件号 ij	01	10	12	21
I_{ij}/I_0	——	1	1	——
l_{ij}/l_0	——	1	1.5	——
k_{ij}	——	1	2/3	——
C_{ij}	——	1	3/4	——
$C_{ij}k_{ij}$	——	1	1/2	——
$\sum C_{ij}k_{ij}$	——	3/2		
λ_{ij}	——	2/3	1/3	——
n_{ij}	——	1/2	0	——

$\bar{M}_{ij}/Ql_0$	-1/10	1/15	0	0
m_{ij}/Ql_0 m'_{ij}/Ql_0	-1/45	-2/45	-1/45	——
M_{ij}/Ql_0	-11/90	1/45	-1/45	0

由表格解出

$M_{01}=-0.1222Ql$

$M_{10}=0.0222Ql$

$M_{12}=-0.0222Ql$

$M_{21}=0$

2. 图 5.5^0

令 $I_{10}=3I_0$， $I_0=I_{12}$，

$l_{10}=l_0$， $l_{12}=l_0$

$q=q_0$， $Q_{10}=q_0l_0$， $Q_{12}={q_0l_0}/{2}$

节点号	0	1		2
杆件号 ij	01	10	12	21
I_{ij}/I_0	——	3	1	——
l_{ij}/l_0	——	1	1	——
k_{ij}	——	3	1	——
C_{ij}	——	1	1	——
$C_{ij}k_{ij}$	——	3	1	——
$\sum C_{ij}k_{ij}$	——	4		
λ_{ij}	——	3/4	1/4	——
n_{ij}	——	1/2	1/2	——
$\bar{M}_{ij}/ql^2$	-1/12	1/12	-11/192	5/192
m_{ij}/ql^2 m'_{ij}/ql^2	-5/512	-5/256	-5/768	-5/1536
M_{ij}/ql^2	-0.0931	0.0638	-0.0638	0.0228

由表格解出：

$M_{01}=-0.0931ql^2$， $M_{10}=-M_{12}=0.0638ql^2$， $M_{21}=0.0228ql^2$

若将图 5.5 中的中间支座去掉，用位移法解之，可有：

$$\begin{cases}16l\theta_2-12\upsilon_2=-\dfrac{5ql^4}{192EI}\\ -12l\theta_2+48\upsilon_2=\dfrac{29ql^4}{32EI}\end{cases}$$

解得：

$$\theta_2=\frac{77ql^3}{96\times52EI}=0.0514\frac{ql^3}{EI},$$

$$\upsilon_2=\frac{227ql^4}{256\times39EI}=0.0227\frac{ql^4}{EI}$$

$M_{12}=-0.140ql^2$，

$M_{23}=0.14ql^2$

$N_{21}=0.040ql$，

$N_{23}=-0.040ql$

5.7 题

计算如表所示

节点号	1	2			3	4
杆件号 ij	12	21	23	24	32	42
I_{ij}/I_0	——	2	3	8	——	——
l_{ij}/l_0	——	1	2.2	3	——	——
k_{ij}	——	2	15/11	8/3	——	——
C_{ij}	——	3/4	3/4	1	——	——
$C_{ij}k_{ij}$	——	3/2	45/44	8/3	——	——
λ_{ij}	——	198/685	297/1507	1056/2055	——	——
n_{ij}	——	0	0	1/2	——	——
$\bar{M}_{ij}/Ql_0$	0	2/15	0	-3.3	0	21/5
m_{ij}/Ql_0 m'_{ij}/Ql_0	0	0.9153	0.6241	1.6273	0	0.8136

M_{ij}/Ql_0	0	1.0487	0.6241	-1.6273	0	5.0136

5.8 题

1）不计 $\overline{45}$ 杆的轴向变形，由对称性知，4、5 节点可视为刚性固定端

2） $Q_{23}=\dfrac{1}{2}q_0\left(3l_0\right)=\dfrac{3}{2}q_0l_0$，$Q_{34}=0.6q_0\left(3l_0\right)=1.8q_0l_0$

$\overline{M_{23}}=Q_{23}(3l_0)/15=\dfrac{3}{10}q_0l_0^2$，　　$\overline{M_{32}}=-Q_{23}(3l_0)/10=-\dfrac{9}{20}q_0l_0^2$

$\overline{M_{34}}=Q_{34}(3l_0)/12=\dfrac{9}{20}q_0l_0^2$

3）计算由下表进行：

$M_{18}=-M_{12}=0.0039q_0l_0^2$，

$M_{21}=-0.0786q_0l_0^2$

$M_{32}=-M_{34}=-0.518q_0l_0^2$，

$M_{25}=-0.0341q_0l_0^2$

$M_{43}=-0.4159q_0l_0^2$，$M_{23}=0.1127q_0l_0^2$

$M_{52}=-0.0170q_0l_0^2$，　其它均可由对称条件得出。

节点号	1		2			3		4	5
杆件号 ij	18	12	21	25	23	32	34	43	52
I_{ij}/I_0	1	1	1	1	6	6	12		
l_{ij}/l_0	6	1	1	3	3	3	3		
k_{ij}	1/6	1	1	1/3	2	2	4		
C_{ij}	1/2	1	1	1	1	1	1		
$C_{ij}k_{ij}$	1/12	1	1	1/3	2	2	4		
$\sum C_{ij}k_{ij}$	13/12		10/3						
λ_{ij}	1/13	12/13	0.3	0.1	0.6	1/3	2/3		
n_{ij}	——	1/2	1/2	1/2	1/2	1/2	1/2		
$\bar{M}_{ij}/ql^2$	0	0	0	0	0.3	−0.45	0.45	−0.45	0
		−.045	−.009	−.003	−.018	−.009			−.015
m_{ij}/ql^2	0.00346	.04154	.02077		.015	.003	.06	.03	
m'_{ij}/ql^2		−.00537	−.01073	−.00358	−.02146	−.01073			−.00179
	.00041	.00496	.00248		.00179	.00358	.00715	.00358	
		−.00064	−.00128	−.00043	−.00256	−.00128			.00022

	.00005	.00059	.00030		.00022	.00043	.00085	.00043	
		-.00008	-.00016	-.00005	-.00031	-.00016			-.00003
					.00003	.00005	.00011	.00006	
			-.00001	-.00000	-.00002	-.00001			
$M_{ij}/q_0 l_0^2$	-0.0039	0.0039	-0.0786	-0.0341	0.1127	-0.5181	0.5181	-0.4159	-0.0170

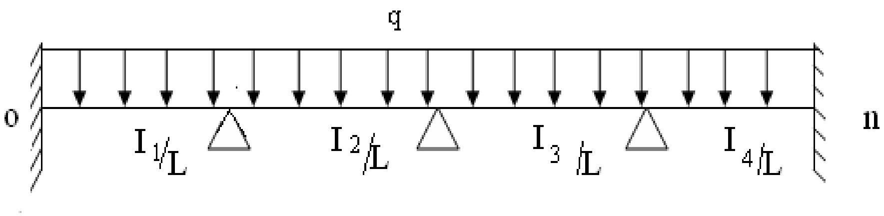


图 5.4a

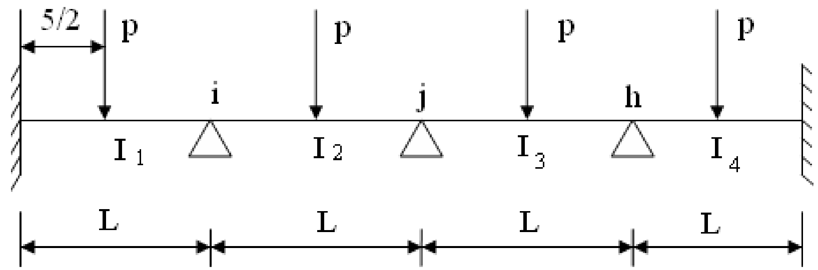


图 5.4b

5.9 题

任一点 i 的不平衡力矩为

$$M_i=\sum_s \bar{M}_{is}=\frac{ql}{12}-\frac{ql}{12}=0$$ （i=1，2，…, h, i, j, …n-1.　s=i-1, i+1）

所以任一中间节点的分配弯矩 m_{ij} 与传导弯矩 $m'_{ij}=n_{ji}m_{ji}$ 均为 0。

任一杆端力矩：$M_{ij}=\bar{M}_{ij}+m_{ij}+m'_{ij}$

$$=M_{ij}-\lambda_{ij}\sum_s \bar{M}_{is}+n_{ji}\left(-\lambda_{ji}\sum_s \bar{M}_{js}\right)=\bar{M}_{ij}\left(0<i<n\right)$$

对两端 $i=0,n$，由于只吸收传导弯矩 $m'_{ij}=0$

$$M_{ij}\underset{\text{两端}}{=}\bar{M}_{ij}+m'_{ij}=\bar{\mathrm{M}}_{\mathrm{ij}}$$

所以对于每个节都有杆端力矩 $M_{ij}=\bar{\mathrm{M}}_{\mathrm{ij}}$

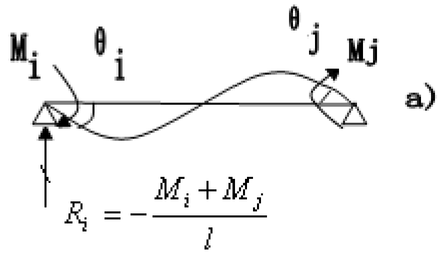
说明：对图 5.4b 所示载荷由于也能使 $\sum M_i=0$，也可以看作两端刚固的单跨梁。

第 6 章 能量法

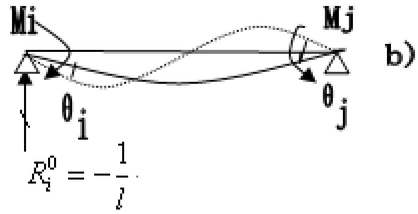
6.1 题

1）方法一 虚位移法

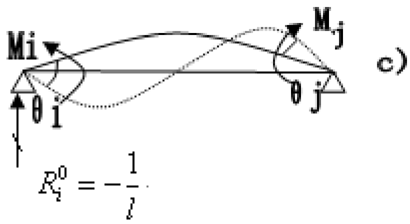
考虑 b),c)所示单位载荷平衡系统，分别给予 a)示的虚变形 ：



$$\frac{M(x)}{EI}dx = \delta d\theta$$



外力虚功为 $\delta W = \begin{Bmatrix} 1\times\theta_\mathrm{i} \\ 1\times\theta_\mathrm{j} \end{Bmatrix}$



虚应变能为

$$\delta\mathrm{V}=\frac{1}{\mathrm{EI}}\int_0^1 \mathrm{M}(x)\mathrm{M}^0(x)\mathrm{d}x$$

$$=\begin{cases} \dfrac{1}{EI}\int_0^l \left(R_i x + M_i\right)\left(R_i^0 x + 1\right)dx \\ \dfrac{1}{EI}\int_0^l \left(R_i x + M_i\right)\left(R_i^0 x\right)dx \end{cases}$$

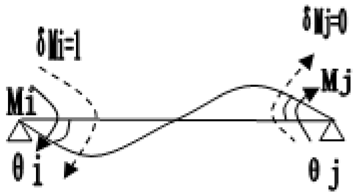
$$=\begin{cases} \dfrac{l}{\mathrm{EI}}\left(\dfrac{\mathrm{M_i}}{3} - \dfrac{\mathrm{M_j}}{6}\right) = \dfrac{l}{3\mathrm{EI}}\left(\mathrm{M_i} - \dfrac{1}{2}\mathrm{M_j}\right) \text{..........b)} \\ \dfrac{l}{\mathrm{EI}}\left(\dfrac{\mathrm{M_j}}{3} - \dfrac{\mathrm{M_i}}{6}\right) = \dfrac{l}{3\mathrm{EI}}\left(\mathrm{M_j} - \dfrac{1}{2}\mathrm{M_i}\right) \text{...........c)} \end{cases}$$

由虚功原理：$\delta W = \delta V$ 得：

$$\begin{Bmatrix} \theta_\mathrm{i} \\ \theta_\mathrm{j} \end{Bmatrix} = \frac{l}{3\mathrm{EI}} \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & 1 \end{bmatrix} \begin{Bmatrix} \mathrm{M_i} \\ \mathrm{M_j} \end{Bmatrix}$$

2）方法二 虚力法（单位虚力法）

$\because$ 梁弯曲应力：$\{\sigma\} = \dfrac{M(x)}{\mathrm{I}}y$



$$\{\varepsilon\} = \frac{\sigma}{\mathrm{E}} = \frac{M(x)}{\mathrm{EI}}y$$

$$M(x) = M_i - \frac{\left(M_i + M_j\right)x}{l}$$

$$\delta M(x)=1-(1+0)\frac{x}{l}$$

给 M_i 以虚变化 $\partial M_i=1$ 虚应力为 $\{\delta\sigma\}=\frac{\delta M(x)}{\mathrm{I}}y$

虚余功：$\delta W^{*}=\theta_{\mathrm{i}}\times 1$

虚余能：$\delta V^{*}=\int_{\Omega}$ （真实应变）×（虚应力）$d\Omega$

$$=\iiint\frac{M(x)}{EI}y\frac{\delta M(x)}{\mathrm{I}}ydxdydz$$

$$=\frac{1}{EI^2}\int_0^l M(x)\delta M(x)\,dx\int_A y^2dA$$

$$=\frac{1}{EI}\int_0^l\left[M_i-\left(M_i+M_j\right)x/l\right](1-x/l)dx$$

$$\therefore\quad Q_i=\frac{l}{3EI}\left(M_i-\frac{1}{2}M_j\right)$$

同理：给 M_j 以虚变化 $\delta M_j=1$，$\left(\delta M_i=0\right)$ 可得（将 i 换为 j）

$$\theta_j=\frac{l}{3EI}\left(-\frac{M_i}{2}+M_j\right)$$

3）方法三 矩阵法（柔度法）

设 $\{\Delta\}=\begin{Bmatrix}\theta_{\mathrm{i}}\\ \theta_{\mathrm{j}}\end{Bmatrix},\{\mathrm{p}\}=\begin{Bmatrix}\mathrm{M_i}\\ \mathrm{M_j}\end{Bmatrix}$，虚力 $\{\delta\mathrm{p}\}=\begin{Bmatrix}\delta M_i\\ \delta M_j\end{Bmatrix},\{\sigma\}=[p]\{\varepsilon\}$

$$\{\sigma\}=\sigma=\frac{M(x)}{I}y=\frac{y}{I}\left[M_i-\left(M_i+M_j\right)x/l\right]=\frac{y}{I}\left[1-\frac{x}{l}\quad-\frac{x}{l}\right]\begin{Bmatrix}M_i\\ M_j\end{Bmatrix}=[c]\{p\}$$

式中 $[c]=\frac{y}{I}\left[\left(1-\frac{x}{l}\right),\left(-\frac{x}{l}\right)\right]$（不妨称为物理矩阵以便与刚度法中几何矩阵 $[B]$ 对应）

虚应力 $\{\delta\sigma\}=[c]\{\delta p\}=[c]\begin{Bmatrix}\delta M_i\\ \delta M_j\end{Bmatrix}$

实应变 $\{\varepsilon\}=[D]^{-1}\{\sigma\}=[D]^{-1}[C]\{p\}$

虚余功 $\delta W^{*}=\{\Delta\}^T\{\delta p\}=\{\delta p\}^T\{\Delta\}=\left(\theta_i\delta M_i+\theta_j\delta M_j\right)$

虚余能 $\delta V^{*}=\int\limits_{\Omega}\{\varepsilon\}^{T}\{\delta\sigma\}d\Omega=\int\limits_{\Omega}\{\varepsilon\sigma\}^{T}\{\varepsilon\}d\Omega$

$$=\int\limits_{\Omega}\{\delta p\}^{T}[C]^{T}[D]^{-1}[C]\{P\}d\Omega=\{\delta p\}^{T}\left[\int\limits_{\Omega}[C]^{T}[D]^{-1}[C]d\Omega\right]\{p\}$$

于虚力原理：$\delta W^{*}=\delta V^{*}$ 考虑到虚力 $\{\delta p\}$ 的任意性。得：

$$\{\Delta\}=\{p\}\int\limits_{\Omega}[C]^{T}[D]^{-1}[C]d\Omega=[\mathrm{A}]\{p\}$$

式中 $[A]=\int\limits_{\Omega}[C]^{T}[D]^{-1}[C]d\Omega$——柔度矩阵（以上推导具有普遍意义）

对本题：

$$[A]=\int\limits_{\Omega}\frac{y}{I}\begin{Bmatrix}1-\dfrac{x}{l}\\-\dfrac{x}{l}\end{Bmatrix}\frac{y}{EI}\left[\left(1-\frac{x}{l}\right)\quad-\frac{x}{l}\right]d\Omega=\frac{1}{EI}\int_{0}^{l}\begin{bmatrix}\left(1-\dfrac{x}{l}\right)^{2} & -\dfrac{x}{l}\left(1-\dfrac{x}{l}\right)\\-\dfrac{x}{l}\left(1-\dfrac{x}{l}\right) & \left(\dfrac{x}{l}\right)^{2}\end{bmatrix}dx$$

$$=\frac{1}{EI}\begin{bmatrix}l/3 & -l/6\\-l/6 & l/3\end{bmatrix}=\frac{l}{3EI}\begin{bmatrix}1 & -1/2\\-1/2 & 1\end{bmatrix}$$

由 $\{\Delta\}=[A]\{p\}$ 展开得：

$$\begin{Bmatrix}\theta_i\\\theta_j\end{Bmatrix}=\frac{l}{3EI}\begin{bmatrix}1 & -1/2\\-1/2 & 1\end{bmatrix}\begin{Bmatrix}M_i\\M_j\end{Bmatrix}$$

6.2 题

<u>方法一 单位位移法</u>

$$\varepsilon=\left(u_j-u_i\right)/l\quad,\quad\sigma=E\varepsilon=E\left(u_j-u_i\right)/l$$

设 $\delta u_i=1$，则 $\delta\varepsilon=-\delta u_i/l=-1/l$

$$T_i\bullet1=\int\limits_{\Omega}\frac{E}{l}\left(u_j-u_i\right)\left(-1/l\right)d\Omega=\frac{-EA}{l^2}\int_0^l\left(u_j-u_i\right)dx=\frac{EA}{l}\left(u_i-u_j\right)$$

同理，令 $\delta u_j=1$ 可得

$$T_j\bullet1=\int\limits_{\Omega}\frac{E}{l}\left(u_j-u_i\right)\left(1/l\right)d\Omega=\frac{EA}{l}\left(u_j-u_i\right)$$

即：$\begin{Bmatrix}T_i\\T_j\end{Bmatrix}=\dfrac{EA}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_i\\u_j\end{Bmatrix}$ 可记为 $\{p_{ij}\}=[K]\{\Delta_{ij}\}$

$[K]$ 为刚度矩阵。

方法二　矩阵虚位移法

设$\left\{p_{ij}\right\}=\left[T_i \quad T_j\right]^T$　　$\left\{\Delta_{ij}\right\}=\left[u_i \quad u_j\right]^T$

$$\because \quad \{\varepsilon\}=\left(u_j-u_i\right)/l=\frac{1}{l}\{-1 \quad 1\}\begin{Bmatrix}u_i\\u_j\end{Bmatrix}\underline{\underline{\Delta}}[B]\left\{\Delta_{ij}\right\}$$

式中　$[B]=\frac{1}{l}\{-1 \quad 1\}$——几何矩阵

$$\therefore \quad \{\sigma\}=[D]\{\varepsilon\}=[D][B]\left\{\Delta_{ij}\right\}$$

设虚位移　$\left\{\delta\Delta_{ij}\right\}=\left[\delta u_i \quad \delta u_j\right]^T$，　虚应变　$\{\delta\varepsilon\}=[B]\left\{\delta\Delta_{ij}\right\}$

外力虚功　$\delta W=\left\{p_{ij}\right\}^T\left\{\delta\Delta_{ij}\right\}=\left\{\delta\Delta_{ij}\right\}^T\left\{p_{ij}\right\}$

虚应变能　$\delta V=\int_\Omega\{\sigma\}^T\{\delta\varepsilon\}\,d\Omega=\int_\Omega\{\delta\varepsilon\}^T\{\sigma\}d\Omega$

$$=\int_\Omega\left\{\delta\Delta_{ij}\right\}^T[B]^T[D][B]\left\{\Delta_{ij}\right\}d\Omega$$

$$=\left\{\delta\Delta_{ij}\right\}^T\left[\int_\Omega[B]^T[D][B]\,d\Omega\right]\left\{\Delta_{ij}\right\}$$

$$\triangleq\left\{\delta\Delta_{ij}\right\}[K]\left\{\Delta_{ij}\right\}$$

由　$\delta W=\delta V$　　得：　$\left\{p_{ij}\right\}=[K]\left\{\Delta_{ij}\right\}$

式中　$[K]=\int_\Omega[B]^T[D][B]\,d\Omega$——刚度矩阵

对拉压杆元　$[K]=EA\int_l\frac{1}{l}\begin{Bmatrix}-1\\1\end{Bmatrix}\frac{1}{l}\{-1 \quad 1\}\,dx=\frac{EA}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$　　详细见方法一。

方法三　　矩阵虚力法

设　$\left\{p_{ij}\right\}=\begin{Bmatrix}T_i\\T_j\end{Bmatrix}$　，　$\left\{\Delta_{ij}\right\}=\begin{Bmatrix}u_i\\u_j\end{Bmatrix}$　，　$\{\delta\}=[D]\{\varepsilon\}$

$$\because \quad \{\sigma\}=\frac{T_j-T_i}{A}=\frac{1}{A}\{-1 \quad 1\}\begin{Bmatrix}T_i\\T_j\end{Bmatrix}\triangleq[C]\left\{p_{ij}\right\}$$

式中　$[C]=\frac{1}{A}[-1 \quad 1]$——物理矩阵（指联系杆端力与应力的系数矩阵）

$\therefore$　$\{\varepsilon\}=[D]^{-1}\{\sigma\}=[D]^{-1}[C]\left\{p_{ij}\right\}$　　虚应力　$\{\delta\sigma\}=[C]\left\{\delta p_{ij}\right\}$

设虚力　$\left\{\delta p_{ij}\right\}=\begin{Bmatrix}\delta T_i\\\delta T_j\end{Bmatrix}$　，　则　$\{\delta\varepsilon\}=[D]^{-1}[C]\left\{\delta p_{ij}\right\}$

虚余功 $\delta W^* = \{\Delta_{ij}\}^T \{\delta p_{ij}\} = \{\delta p_{ij}\}^T \{\Delta_{ij}\}$

虚余能 $\delta V^* = \int_\Omega \{\varepsilon\}^T \{\delta\sigma\} d\Omega = \int_\Omega \{\delta\sigma\}^T \{\varepsilon\} d\Omega$

$$= \int_\Omega \{\delta p_{ij}\}^T [C]^T [D]^{-1} [C] \{p_{ij}\} d\Omega$$

$$= \{\delta p_{ij}\} \left[\int_\Omega [C]^T [D]^{-1} [C] d\Omega \right] \{p_{ij}\}$$

$$\triangleq \{\delta p_{ij}\} [A] \{p_{ij}\}$$

式中 $[A] = \int_\Omega [C]^T [D]^{-1} [C] d\Omega$ ——柔度矩阵

对拉压杆： $[K] = \frac{A}{E} \int_l \frac{1}{A} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \frac{1}{A} \{-1 \quad 1\} dx = \frac{l}{EA} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$\therefore \quad \{\Delta_{ij}\} = [A]\{p_{ij}\}$

即 $\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \frac{l}{EA} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$

讨论： 比较方法二、三。

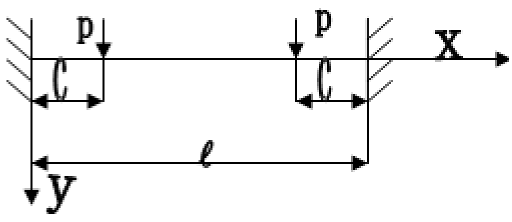
结论： $\{p_{ij}\} = [K]\{\Delta_{ij}\}$， $\{\Delta_{ij}\} = [A]\{p_{ij}\}$

若 $[K]$与$[A]$的逆矩阵存在（遗憾的是并非总是存在），则，$[K]^{-1}$

实际上是一个柔度矩阵，$[A]^{-1}$实际上是一个刚度矩阵

6.3 题

1）6.3^{0}如图所示



设 $v(x) = \sum_{n=1}^{\infty} a_n \left(1 - \cos\frac{2n\pi x}{l}\right)$

显然满足$x=0, x=l$处的

变形约束条件

$$v(0) = v(l) = 0$$
$$v'(0) = v'(l) = 0$$

变形能 $V = \frac{EI}{2}\int_0^l (v'')^2 dx = \frac{EI}{2}\int_0^l \left[\sum_{n=1}^{\infty} a_n \left(\frac{2n\pi}{l}\right)^2 \cos\frac{2n\pi x}{l} \right]^2 dx$

$$=\frac{EI}{2}\sum_{n=1}^{\infty}a_n^2\left(\frac{2n\pi}{l}\right)^4\frac{l}{2}$$

力函数 $Ш=pv(c)+pv(l-c)=2pv(c)$（对称）

$$=2p\sum_{n=1}^{\infty}a_n\left(1-\cos\frac{2n\pi c}{l}\right)$$

由 $\frac{\partial(V-Ш)}{\partial a_n}=0$，所以 $\frac{\partial V}{\partial a_n}=\frac{\partial Ш}{\partial a_n}$。即

$$\frac{EIl}{2}a_n\left(\frac{2n\pi}{l}\right)^4=2p\left(1-\cos\frac{2n\pi c}{l}\right)$$

所以，$a_n=\frac{pl^3}{4EI\pi^4}\cdot\frac{\left(1-\cos\frac{2n\pi c}{l}\right)}{n^4}$

$$v(x)=\frac{pl^3}{4\pi^4EI}\sum_{n=1}^{\infty}\frac{1}{n^4}\left(1-\cos\frac{2n\pi c}{l}\right)\left(1-\cos\frac{2n\pi x}{l}\right)$$

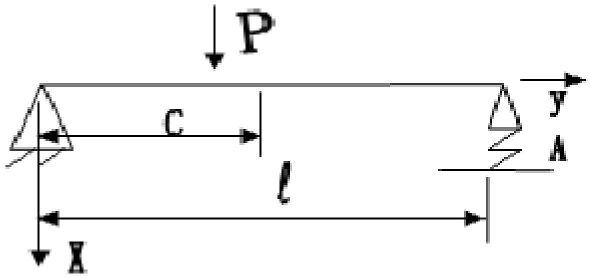
2）6.4^0 如图所示

设 $v(x)=a_0x+\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{l}$

$$V=\frac{EI}{2}\int_0^l\left[-\sum_{n=1}^{\infty}a_n\left(\frac{n\pi}{l}\right)^2\sin\frac{n\pi x}{l}\right]^2dx+\frac{[v(l)]^2}{2A}=\frac{EI}{2}\sum_{n=1}^{\infty}a_n^2\left(\frac{n\pi}{l}\right)^4\frac{l}{2}+\frac{(a_0l)^2}{2A}$$

$$Ш=pU(c)=p\sum_{n=1}^{\infty}a_n\sin\frac{n\pi c}{l}+a_0pc$$

由 $\frac{\partial(V-Ш)}{\partial a_0}=0$



得 $a_0l^2/A=pc$，

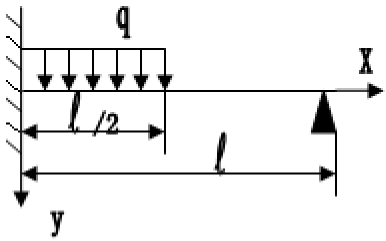
所以，$a_0=Apc/l^2$

由 $\frac{\partial(V-Ш)}{\partial a_n}=0$，得

$$\frac{EIl}{2}\left(\frac{n\pi}{l}\right)^4 a_n = p\sin\frac{n\pi c}{l} \quad 所以，\quad a_n = \frac{2pl^3}{EI(n\pi)^4}\sin\frac{n\pi c}{l}$$

$$\therefore \quad v(x) = \frac{Apc}{l^2}x + \frac{2pl^3}{EI\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

3）6.5^{0}如图所示　令 $v(x) = ax^2(l-x)$



所以，$$V = \frac{EI}{2}\int_0^l v''^2 dx$$

$$= \frac{EI}{2}\int_0^l (2al-6ax)^2 dx$$

$$= 2a^2 EIl^3$$

$$Ш = \int_0^{l/2} qU(x)dx = \int_0^{l/2} qax^2(l-x)dx = \frac{5}{192}qal^4$$

由 $\frac{\partial(V-Ш)}{\partial a} = 0$　得 $4aEIl^3 = \frac{5}{192}ql^4$　所以，$a = \frac{5ql}{768EI}$

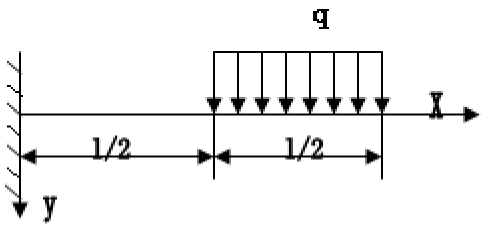
$$\therefore \quad v(x) = \frac{5ql}{768EI}x^2(l-x)$$

4）6.6^{0}所示如图，

设 $v(x) = a_1x^2 + a_2x^3$，$v(x)'' = 2(a_1 + 3a_2x)$

$$V = \frac{EI}{2}\int_0^l v''^2 dx = \frac{EI}{2}\int_0^l 4(a_1+3a_2x)^2 dx$$

$$= 2EIl\left(a_1^2 + 3a_1a_2l + 3a_2^2l^2\right)$$



$$Ш = \int_{l/2}^{l} qv(x)dx = q\int_{l/2}^{l}\left(a_1x^2 + a_2x^3\right)dx$$

$$= \frac{ql^3}{8}\left(\frac{7a_1}{3} + \frac{15a_2l}{8}\right)$$

由 $\frac{\partial(V-Ш)}{\partial a_1} = 0$　得 $2EIl(2a_1 + 3a_2l) = 7ql^3/24$

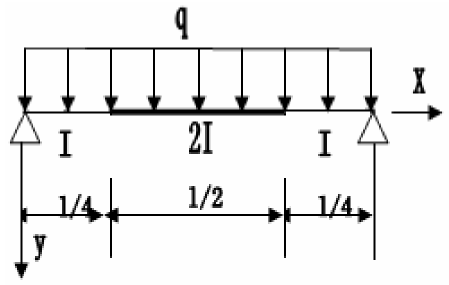
由 $\frac{\partial(V-Ш)}{\partial a_2} = 0$　得 $6EIl\left(a_1l + 2a_2l^2\right) = 15ql^4/64$

解上述两式得 $\begin{cases} a_1 = \dfrac{67ql^2}{384EI} \\ a_2 = \dfrac{-13ql}{192EI} \end{cases}$

$$\therefore \quad v(x)=0.1745\frac{ql^2}{EI}x^2-0.0677\frac{ql}{EI}x^3$$

6.4 题

如图所示



设 $v(x)=a_1\sin\frac{\pi x}{l}$

$$V=2\left[\frac{EI}{2}\int_0^{l/4}v''^2dx+\frac{E(2I)}{2}\int_{l/4}^{l/2}v''^2dx\right]$$

$$=EI\int_0^{l/4}a_1^2\left(\frac{\pi}{l}\right)^4\left(\sin\frac{\pi x}{l}\right)^2dx+2EI\int_{l/4}^{l/2}a_1^2\left(\frac{\pi}{l}\right)^4\left(\sin\frac{\pi x}{l}\right)^2dx$$

$$=EIa_1^2\left(\frac{\pi}{l}\right)^4\frac{l}{4}\left(\frac{3}{2}+\frac{1}{\pi}\right)$$

$$\amalg=\int_0^l qv(x)\,dx=q\int_0^l a_1\sin\frac{\pi x}{l}dx=2qla_1/\pi$$

由 $\frac{\partial(V-\amalg)}{\partial a_1}=0$ 得 $EIa_1\left(\frac{\pi}{l}\right)^4\frac{l}{2}\left(\frac{3}{2}+\frac{1}{\pi}\right)=\frac{2ql}{\pi}$

所以， $a_1=\frac{4}{\pi^5\left(\frac{3}{2}+\frac{1}{\pi}\right)}\frac{ql^4}{EI}=0.00718\frac{ql^4}{EI}$

$$U(x)=0.00718\frac{ql^4}{EI}\sin\frac{\pi x}{l}$$

6.5 题

如图所示

设 $v(x)=\sum_{n=1}^{\infty}a_n\sin\frac{(2n-1)\pi x}{2l}$

$$V=\frac{EI}{2}\int_0^{2l}\left[v(x)''\right]^2dx+\frac{\left[v(l)\right]^2}{2A}$$

$$=\frac{EI}{2l^3}\left[\sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi}{2}\right)^4a_n^2+\left(\sum_{n=1}^{\infty}a_n\sin\left(\frac{2n-1}{2}\pi\right)\right)^2\right]$$

其中， $A=\dfrac{l^3}{EI}$

$$\frac{\partial V}{\partial a_n}=\frac{EI}{l^3}\left(\frac{2n-1}{2}\pi\right)^4 a_n+\frac{EI}{l^3}\left(\sum_{n=1}^{\infty}a_n\sin\left(\frac{2n-1}{2}\pi\right)\right)\bullet\sin\left[\frac{(2n-1)\pi}{2}\right]$$

$$\text{Ц}=\int_0^{2l}qv(x)\,dx=q\int_0^{2l}\sum_{n=1}^{\infty}a_n\sin\frac{(2n-1)\pi x}{2l}dx$$

$$=q\sum_{n=1}^{\infty}a_n\left(\frac{2l}{(2n-1)\pi}\right)\left[1-\cos(2n-1)\pi\right]=\frac{4ql}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{2n-1}$$

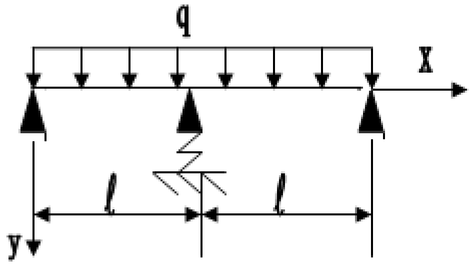
所以， $\dfrac{\partial \text{Ц}}{\partial a_n}=\dfrac{4ql}{(2n-1)\pi}$

取前两项得 $\dfrac{\partial V}{\partial a_1}=\dfrac{EI}{l^3}\left(\dfrac{\pi}{2}\right)^4 a_1+\dfrac{EI}{l^3}(a_1-a_2)$， $\dfrac{\partial V}{\partial a_2}=\dfrac{EI}{l^3}\left(\dfrac{3\pi}{2}\right)^4 a_2-\dfrac{EI}{l^3}(a_1-a_2)$

由 $\dfrac{\partial(V-\text{Ц})}{\partial a_1}=0$ 得 $\left\{\dfrac{EI}{l^3}\left[\left(\dfrac{\pi}{2}\right)^4+1\right]\right\}a_1-\dfrac{EI}{l^3}a_2=\dfrac{4ql}{\pi}$

由 $\dfrac{\partial(V-\text{Ц})}{\partial a_2}=0$ 得 $\left\{\dfrac{EI}{l^3}\left[\left(\dfrac{3\pi}{2}\right)^4+1\right]\right\}a_2-\dfrac{EI}{l^3}a_1=\dfrac{4ql}{3\pi}$

即： $\begin{cases}7.088a_1-a_2=\dfrac{4ql^4}{\pi EI}\\ a_1-494.133a_2=-\dfrac{4ql^4}{3\pi EI}\end{cases}$



解得 $\begin{cases}a_1=0.1798\dfrac{ql^4}{EI}\\ a_2=0.00118\dfrac{ql^4}{EI}\end{cases}$

$$\therefore\ v(x)=\left(0.180\sin\frac{\pi x}{2l}+0.0012\sin\frac{3\pi x}{2l}\right)\frac{ql^4}{EI}$$

$\therefore$ 中点挠度 $v\left(\dfrac{l}{2}\right)=0.1786\dfrac{ql^4}{EI}$

6.6 题 取 $v_1(x)=\sum a_n \sin\frac{n\pi x}{l}, v_2(x)=\sum b_n \sin\frac{n\pi x}{l}$

$$V=\frac{EI}{2}\int_0^l v_1''^2 dx+\frac{GA_s}{2}\int_0^l v_2'^2 dx$$

$$=\frac{EI}{2}\int_0^l\left[\sum -a_n\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}\right]^2 dx+\frac{GA_s}{2}\int_0^l\left[\sum b_n\left(\frac{n\pi}{l}\right)\cos\frac{n\pi x}{l}\right]^2 dx$$

$$=\frac{EI}{2}\sum a_n^2\left(\frac{n\pi}{l}\right)^4\frac{l}{2}+\frac{GA_s}{2}\sum b_n^2\left(\frac{n\pi}{l}\right)^2\frac{l}{2}$$

$$=\frac{EIl}{4}\sum\left(\frac{n\pi}{l}\right)^4 a_n^2+\frac{GA_s l}{4}\sum\left(\frac{n\pi}{l}\right)^2 b_n^2$$

$$\frac{\partial V}{\partial a_n}=\frac{EIl}{2}(\frac{n\pi}{l})^4 a_n,\frac{\partial V}{\partial b_n}=\frac{GA_s l}{2}(\frac{n\pi}{l})^2 b_n$$

$$\text{Ц}=\int_0^l q v_1 dx+\int_0^l q v_2 dx$$

$$=q\int_0^l\sum a_n \sin\frac{n\pi x}{l}dx+q\int_0^l\sum b_n \sin\frac{n\pi x}{l}dx$$

$$=q\sum a_n\left(\frac{n\pi}{l}\right)^{-1}(1-\cos n\pi)+q\sum b_n\left(\frac{n\pi}{l}\right)^{-1}(1-\cos n\pi)$$

$$\therefore \partial\text{Ц}/\partial a_n=q\left(\frac{l}{n\pi}\right)(1-\cos n\pi),\partial\text{Ц}/\partial b_n=q\left(\frac{l}{n\pi}\right)(1-\cos n\pi)$$

由 $\frac{\partial(V-\text{Ц})}{\partial a_n}=0$ 得 $a_n=\frac{2ql^4(1-\cos n\pi)}{(n\pi)^5 EI}\overset{n\text{为奇数}}{=}\frac{4ql^4}{(n\pi)^5 EI}$

由 $\frac{\partial(V-\text{Ц})}{\partial a_n}=0$ 得 $b_n=\frac{2ql^2(1-\cos n\pi)}{(n\pi)^3 GA_s}\overset{n\text{为奇数}}{=}\frac{4ql^2}{(n\pi)^3 GA_s}$

$$\therefore U(x)=U_1(x)+U_2(x)$$

$$=\frac{4ql^4}{\pi^5 EI}\sum_n\frac{1}{n^5}\sin\frac{n\pi x}{l}+\frac{4ql^2}{\pi^3 GA_S}\sum_n\frac{1}{n^3}\sin\frac{n\pi x}{l}$$

$(N=1,3,5,\cdots\cdots)$

6.7 题

1）图 6.9 对于等断面轴向力沿梁长不变时，复杂弯曲方程为：

$$EIV^{IV}-TV''-q=0$$

取 $v(x)=\sum_n a_n \sin\frac{n\pi x}{l}$ 能满足梁段全部边界条件

$x=0,l$处 $v=0, v' \neq 0, v'' = 0, v''' \neq 0 \therefore \int_0^l (EIV^{IV} - TV'' - q)qvdx = 0$

$$\therefore 有 \int_0^l \left[EI \sum a_n (\frac{n\pi}{l})^4 \sin\frac{n\pi x}{l} - T \sum a_n (\frac{n\pi}{l})^2 (-\sin\frac{n\pi x}{l}) - q \right] \sin\frac{n\pi x}{l} dx = 0$$

积分：$EIa_n \left(\frac{n\pi}{l}\right)^4 \frac{l}{2} + Ta_n \left(\frac{n\pi}{l}\right)^2 \frac{l}{2} - q\left(\frac{l}{n\pi}\right)\left[-\cos\frac{n\pi x}{l}\right]_0^l = 0$

即：$a_n = \dfrac{q\left(\frac{l}{n\pi}\right)(1-\cos n\pi)}{\frac{EIl}{2}\left(\frac{n\pi}{l}\right)^4 + T\left(\frac{n\pi}{l}\right)^2 \frac{l}{2}} = \begin{cases} 0(n为偶数) \\ \dfrac{4ql^4}{EI(n\pi)^5\left[1+4u^2/(n\pi)^2\right]}(n为奇数) \end{cases}$

式中：$u = \frac{l}{2}\sqrt{T/EI}$ 今已知 u=1

$$\therefore v(x) = \frac{4ql^4}{EI\pi^5} \sum_N \frac{\sin\frac{n\pi x}{l}}{n^5(1+4u^2/\pi^2 n^2)} (n = 1,3,5 \cdots)$$

$$\therefore v(\frac{l}{2}) \overset{取一项}{\underset{n=1}{=}} \frac{4ql^4}{EI(\pi n)^5(1+4u^2/\pi^2 n^2)} = 0.009301 ql^4/EI$$

准确解为：$v\left(\frac{l}{2}\right) = \frac{5ql^4}{384EI} \cdot f_0(1) = \left[\frac{5}{384} \times 0.711\right] ql^4/EI = 0.009258 ql^4/EI$

误差仅为 0.46%

结论：1）引进 $T_{cr} = (n\pi)^2 EI/l^2$——单跨简支压杆临界力

$$u^2 = \frac{l^2}{4}\left(T/EI\right), 4/\pi^5 \approx \frac{5}{384}$$

2）取一项，中点挠度表达式可写成如下讨论的形式：

$$v\left(\frac{l}{2}\right) = \frac{5ql^4}{EI384}\left[\frac{1}{1 \pm \left|T/T_{cr}\right|}\right] = \begin{cases} \dfrac{5}{384}\dfrac{ql^4}{EI}(T=0) \\ \infty(失稳)(T = T_{cr}的压力时) \end{cases}$$

式中：当 T 为拉力时取正号（此时相当一缩小系数，随 T↑而↓）≤1
当 T 为压力时取负号（此时相当一放大系数，随 T↑而↑）≥1

2）图 6.10∵弹性基础梁平衡方程为：$EIV^{IV} + kv - q = 0$

$$\therefore \int_0^l \left[EIV^{IV} + kv - q\right]\delta V dx = 0$$

取：$V(x)=\sum_n a_n \sin\frac{n\pi x}{l}$代入上式：

$$\delta a_n \int_0^l \left[EI \sum_n a_n \left(\frac{n\pi}{l}\right)^4 \sin\frac{n\pi x}{l} + k\sum_n a_n \sin\frac{n\pi x}{l} - q \right] \sin\frac{n\pi x}{l} dx = 0$$

由于δa_n的随意性有式中积分为 0，即：

$$EIa_n\left(\frac{n\pi}{l}\right)^4 \frac{l}{2} + ka_n\frac{l}{2} - q\left(\frac{l}{n\pi}\right)(1-\cos n\pi) = 0$$

$$\therefore a_n = \frac{q\left(\frac{l}{n\pi}\right)(1-\cos n\pi)}{\frac{EIl}{2}\left(\frac{n\pi}{l}\right)^4 + \frac{kl}{2}} = \frac{4ql^4}{EI(n\pi)^5\left[1+k/EI(\frac{n\pi}{5})^4\right]}(n\text{为奇数})$$

由$u=\frac{l}{2}\sqrt[4]{k/4EI}$得$k=\left(\frac{2u}{l}\right)^4\bullet 4EI$代入得

$$a_n = \frac{4ql^4}{EI(n\pi)^5\left[1+4\left(2u/n\pi\right)^4\right]}$$

$$v(x) = \left(\frac{4ql^4}{EI\pi^5}\right)\sum_n \frac{\sin\frac{n\pi x}{l}}{n^5\left[1+\frac{k}{EI(n\pi/l)^4}\right]}(n=1,3,5\bullet\bullet\bullet)$$

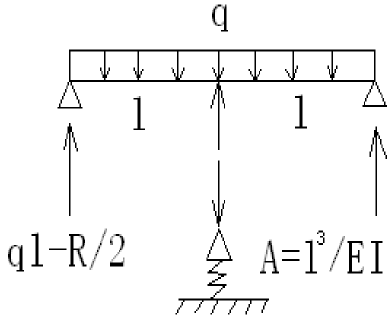
今取一项，且令 u=1，求中点挠度

$$v(\frac{l}{2}) = \left[\frac{4}{\pi^5\left[1+4\left(2/\pi\right)^4\right]}\right]\frac{ql^4}{EI} = 0.007888\, ql^4/EI$$

准确值：$v(\frac{1}{2}) = \frac{q}{k}\left[1-\varphi_0(u)\right] = \left[\frac{1-0.448}{4\bullet(2\times 1)^4}\right]\frac{ql^4}{EI} = 0.008625\, ql^4/EI$

误差为 8.5%误差较大，若多取几项，如取二项则误差更大，∴交错级数的和小于首项，即$v\left(\frac{l}{2}\right)$按级数法只能收敛到略小于精确解的一个值，此矛盾是由于φ_0是近似值。

6.8 题



$$v = v(梁) + v(支)$$
$$= \frac{2}{EI}\int_0^l \frac{M^2(x)}{2}dx + \frac{1}{2}AR^2$$

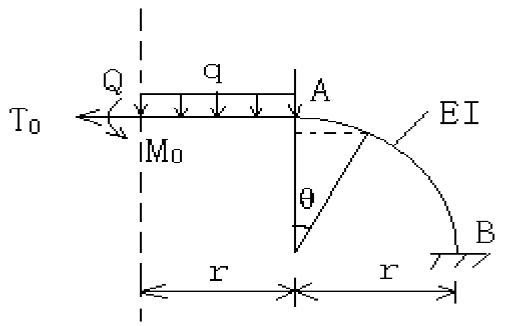
$$\frac{\partial v}{\partial R} = \frac{2}{EI}\int_0^l M(x)\frac{\partial M}{\partial R}dx + AR$$
$$= \frac{2}{EI}\int_0^l \left[\left(ql - \frac{R}{2}\right)x - \frac{qx^2}{2}\right](-\frac{x}{2})dx + AR$$
$$= \frac{2}{EI}\left[\left(ql - \frac{R}{2}\right)\left(\frac{-l^3}{3\times 2}\right) + \frac{ql^4}{16}\right] + \frac{l^3}{EI}R$$
$$= \frac{2}{EI}\left[\left(-\frac{1}{6} + \frac{1}{16}\right)ql^4\right] + \frac{l^3}{EI}(\frac{1}{6} + 1)$$

由最小功原理：$\partial v / \partial R = 0$ 解出：$R = 5ql / 28$

$$\therefore \; v_{中} = 5q(2l)^4 / 384EI - R(2l^3) / 48EI$$
$$= 5ql^4 / 28EI \approx 0.1785\,ql^4 / EI$$

6.9 题

由对称性可知，对称断面处剪力为零，转角$\theta_0 = 0$，静不定内力T_0和M_0可最小功原理求出：



$$M(s) = \begin{cases} M_0 + qs_1^2 / 2 \text{ —}(OA段) \\ (M_0 + qr^2/2) + 2qr^2\sin\theta + T_0 r(1-\cos\theta)\text{—}(AB段) \end{cases}$$

$$\frac{\partial M(s)}{\partial M_0} = \begin{cases} 1 & (OA段) \\ 1 & (AB段) \end{cases} \qquad \frac{\partial M(s)}{\partial T_0} = \begin{cases} 0 & \text{—}(OA段) \\ r(1-\cos\theta) & \text{—}(AB段) \end{cases}$$

最小功原理：

$$\frac{\partial V}{\partial M_0}=\int_s \frac{M(s)}{EI}\frac{\partial M(s)}{\partial M_0}ds$$
$$=\frac{1}{EI}\int_0^r\left(M_0+{qs_1^2}/{2}\right)ds_1+\frac{1}{EI}\int_0^{\frac{\pi}{2}}\left[\left(M_0+qr^2/2\right)+2qr^2\sin\theta+T_0r\left(1-\cos\theta\right)\right]rd\theta$$
$$=0$$

$$\frac{\partial V}{\partial T}=\frac{1}{EI}\int_0^{\pi/2}\left[M_0+{qr^2}/{2}+2qr^2\sin\theta+T_0r(1-\cos\theta)\right]r\left(1-\cos\theta\right)\bullet rd\theta=0$$

分别得：$\begin{cases}M_0\left(1+{\pi}/{2}\right)+T_0r\left({\pi}/{2}-1\right)=-qr^2\left(2+\dfrac{1}{6}+\dfrac{\pi}{4}\right)\\ M_0\left(1-{\pi}/{2}\right)+T_0r\left(2-{3\pi}/{4}\right)=qr^2\left(\dfrac{1}{2}+\dfrac{\pi}{4}\right)\end{cases}$

解得：$\begin{cases}M_0=-0.5388qr^2\\ T_0=-2.7452qr\end{cases}$ $\therefore M(s)$表达式正确

由$\dfrac{\partial M}{\partial s_1}=0$ 　　得极值点在 $s_t=0$ 点,该处极值为 $M_1=M_0$

由$\dfrac{\partial M}{\partial s_2}=0$ 　　得 $tg\theta=-{2qr}/{T_0}=0.7285,\theta\approx0.6296$

极值为 $M_2=\left[\left(-0.5388+\dfrac{1}{2}\right)qr^2+2qr^2\sin0.6296+\left(-2.7452qr^2\right)\left(1-\cos\theta\right)\right]$
$$=0.61qr^2$$
区间端点 B 处
$$M_B=\left[\left(-0.5388+\frac{1}{2}\right)qr^2+2qr^2\sin\frac{\pi}{2}-\left(2.7452qr^2\right)\bullet1\right]=-0.79qr^2$$

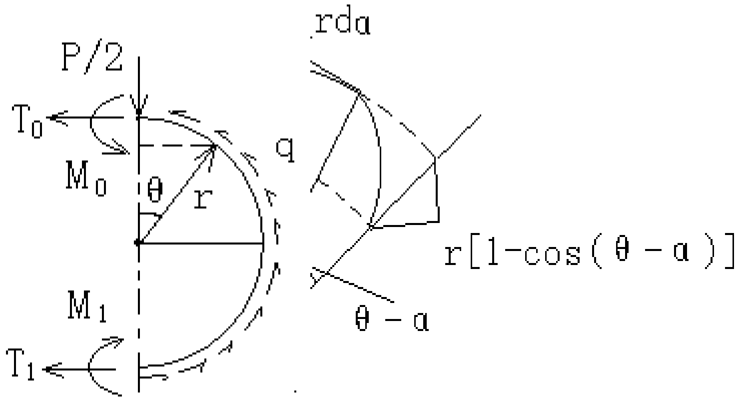
$\therefore\left|M_{\max}\right|=\max\left\{\left|M_0\right|,\left|M_1\right|,\left|M_B\right|\right\}=\left|M_B\right|$
$\therefore M_{\max}=M_B=-0.79qr^2$(发生在支撑处)

6.10 题

由左右对称, ∴对陈断面 01 上无剪力。

有垂向静力平衡条件：$\int_0^{\pi}qr\sin\theta d\theta={P}/{2}$

解得：$q=P/4r$



任意断面弯矩为：

$$M(s)=M_0+\frac{\Pr}{2}\sin\theta+T_0r(1-\cos\theta)+\int_0^\theta qr^2\left[-1+\cos\left(\theta-\alpha\right)\right]d\alpha$$

$$=M_0+T_0r(1-\cos\theta)+\frac{\Pr}{2}\sin\theta+qr^2\left(-\sin\theta+\theta\right)$$

$$\frac{\partial M}{\partial M_0}=1,\frac{\partial M}{\partial T_0}=r\left(1-\cos\theta\right)$$

有最小功原理确定 T_0 和 M_0

$$\frac{\partial V}{\partial M_0}=\frac{1}{EI}\int_0^\pi\left[M_0+T_0r(1-\cos\theta)+\frac{\Pr}{2}\sin\theta+qr^2(-\sin\theta+\theta)\right]rd\theta=0$$

即：$M_0\pi+T_0\pi r+\Pr+qr^2(-2+\pi^2/2)=0$

$$\frac{\partial V}{\partial T_0}=\frac{1}{EI}\int_0^\pi\left[M_0+T_0r(1-\cos\theta)+\frac{\Pr}{2}\sin\theta+qr^2(-\sin\theta+\theta)\right]r(1-\cos\theta)rd\theta=0$$

即$\int_0^\pi M(s)(1-\cos\theta)d\theta=0-\int_0^\pi M(s)\cos\theta d\theta=0$

$$\therefore\int_0^\pi\left[(M_0+T_0r)\cos\theta-T_0r\cos^2\theta+\frac{\Pr}{2}\sin\theta+qr^2(-\sin\theta\cos\theta+\theta\cos\theta)\right]d\theta=0$$

得：$-T_0r\pi/2-2qr^2=0\therefore T_0=-4qr/\pi=-P/\pi$(与图中假设$T_0$方向相反)

$$\therefore M_0=\Pr(4-\pi^2)/8\pi$$

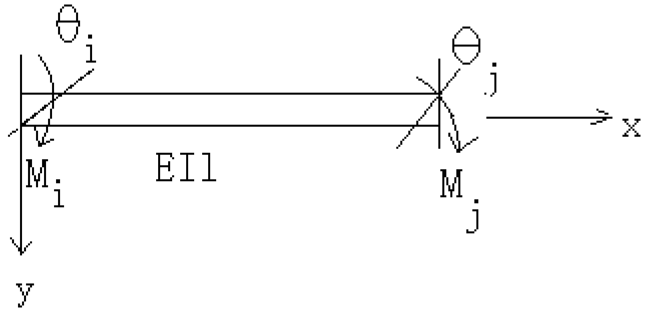
$$\therefore M(s)=\frac{\Pr}{8\pi}(4-\pi^2)-\frac{\Pr}{\pi}(1-\cos\theta)+\frac{\Pr}{4}\sin\theta-\frac{\Pr}{4}\theta$$

$$=\left[\frac{4-\pi^2}{8\pi}-\frac{1}{\pi}+\frac{\cos\theta}{\pi}+\frac{\sin\theta}{4}-\frac{\theta}{4}\right]\Pr$$

第 7 章 矩阵法

7.1 题

解：由 ch2/2.4 题/2.6 图计算结果



$$\nu=\theta_1 x-\frac{2\theta_1+\theta_2}{l}x^2+\frac{\theta_1+\theta_2}{l^2}x^3$$

$$\nu'(x)=\theta_1-\frac{2\theta_1+\theta_2}{l}x+3\frac{\theta_1+\theta_2}{l^2}x^2,\quad \nu''(x)=-\frac{2\theta_1+\theta_2}{l}+6\frac{\theta_1+\theta_2}{l^2}x$$

$$\because \varepsilon=y\nu''=y\left[\left(\frac{-4}{l}+\frac{6x}{l^2}\right)\left(\frac{-2}{l}+\frac{6x}{l^2}\right)\right]\begin{Bmatrix}\theta_i\\ \theta_j\end{Bmatrix}$$

$$\therefore [B]=\frac{2y}{l}\left[\left(\frac{3x}{l}-2\right)\left(\frac{3x}{l}-1\right)\right],[D]=E$$

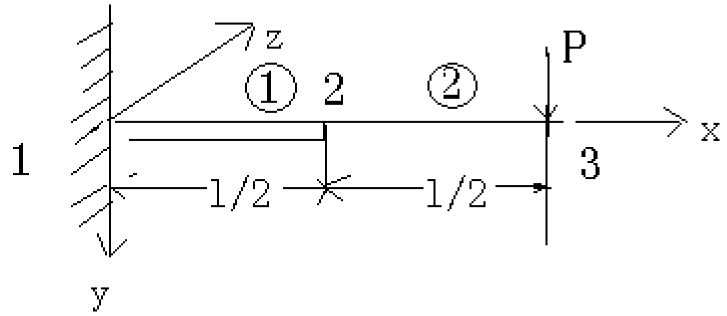
$$\left[K^e\right]=\int_\Omega [B]^T[D][B]d\Omega=\int_\Omega \frac{4y^2}{l^2}\begin{bmatrix}\frac{3x}{l}-2\\ \frac{3x}{l}-1\end{bmatrix}[E]\left[\left(\frac{3x}{l}-2\right)\quad\left(\frac{3x}{l}-1\right)\right]d\Omega$$

$$=\frac{4EI}{l^2}\int_0^l\begin{bmatrix}\left(\frac{3x}{l}-2\right)^2 & \left(\frac{3x}{l}\right)^2-3\frac{3x}{l}+2\\ \text{对称} & \left(\frac{3x}{l}-1\right)^2\end{bmatrix}=\frac{4EI}{l^2}\begin{bmatrix}l & \frac{l}{2}\\ \frac{l}{2} & l\end{bmatrix}=\frac{2EI}{l}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

7.2 题

解：如图示离散为

3 个节点，2 个单元



$$\left[K^{(1)}\right]=\frac{E(2I)}{\left(\frac{l}{2}\right)}\begin{bmatrix} \frac{12}{\left(\frac{l}{2}\right)^2} & \frac{6}{\left(\frac{l}{2}\right)} & -\frac{12}{\left(\frac{l}{2}\right)^2} & \frac{6}{\left(\frac{l}{2}\right)} \\ \frac{6}{\left(\frac{l}{2}\right)} & 4 & -\frac{6}{\left(\frac{l}{2}\right)} & 2 \\ -\frac{12}{\left(\frac{l}{2}\right)^2} & -\frac{6}{\left(\frac{l}{2}\right)} & \frac{12}{\left(\frac{l}{2}\right)^2} & -\frac{6}{\left(\frac{l}{2}\right)} \\ \frac{6}{\left(\frac{l}{2}\right)} & 2 & -\frac{6}{\left(\frac{l}{2}\right)} & 4 \end{bmatrix}=\begin{bmatrix} K_{11}{}^{(1)} & K_{12}{}^{(1)} \\ K_{21}{}^{(1)} & K_{22}{}^{(1)} \end{bmatrix}=2\left[K^{(2)}\right]$$

$$=\begin{bmatrix} K_{22}{}^{(2)} & K_{23}{}^{(2)} \\ K_{32}{}^{(2)} & K_{33}{}^{(2)} \end{bmatrix}$$

形成$[K]\begin{bmatrix} K_{11}{}^{(1)} & K_{12}{}^{(1)} & [0] \\ K_{21}{}^{(1)} & K_{22}{}^{(1)}+K_{22}{}^{(2)} & K_{23}{}^{(2)} \\ [0] & K_{32}{}^{(2)} & K_{33}{}^{(2)} \end{bmatrix}\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix}$

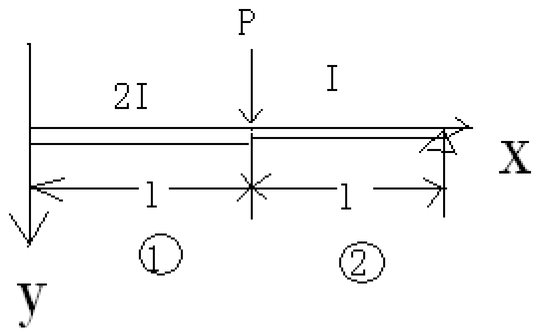
将各子块代入得：

$$\frac{EI}{(l/2)}\begin{bmatrix} \frac{24}{(l/2)^2} & \frac{6x^2}{(l/2)} & \frac{-24}{(l/2)^2} & \frac{12}{(l/2)} & & \\ \frac{6x^2}{(l/2)} & 4\times 2 & \frac{12}{(l/2)} & 4 & & \\ \frac{-24}{(l/2)^2} & \frac{-12}{(l/2)} & \frac{36}{(l/2)^2} & \frac{-6}{(l/2)} & \frac{24}{(l/2)^2} & \frac{-6}{(l/2)} \\ \frac{12}{(l/2)} & 4 & \frac{-6}{(l/2)} & 12 & \frac{-6}{(l/2)} & 2 \\ & & \frac{24}{(l/2)^2} & \frac{-6}{(l/2)} & \frac{24}{(l/2)^2} & \frac{-6}{(l/2)} \\ & & \frac{-6}{(l/2)} & 2 & \frac{-6}{(l/2)} & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \\ v_3 \\ \theta_{z3} \end{Bmatrix}=\begin{Bmatrix} R_{y1} \\ M_{R1} \\ 0 \\ 0 \\ P \\ 0 \end{Bmatrix}$$

划去 1、2 行列，（$\because v_1=\theta_{z1}=0$）约束处理后得：

$$\frac{2EI}{l}\begin{bmatrix} 144/l^2 & \frac{-12}{l} & -48/l^2 & \frac{12}{l} \\ \frac{-12}{l} & 12 & \frac{-12}{l} & 2 \\ -48/l^2 & \frac{-12}{l} & 48/l^2 & \frac{-12}{l} \\ \frac{12}{l} & 2 & \frac{-12}{l} & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_{z2} \\ v_3 \\ \theta_{z3} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix}$$

图 7.3 离散如图



∵杆元尺寸图 7.2（以 2l 代 l），∴$\left[K^e\right]$不变，离散方式一样，组装成的整体刚度矩一样$\left[K\right]$

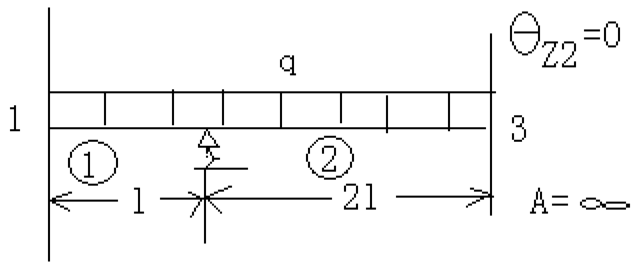
$\left\{P\right\}^T=\left\{R_{1y}\quad M_{R1}\quad P\quad 0\quad R_{3y}\quad 0\right\}^T$

$\left\{\delta\right\}^T=\left\{\nu_1\quad \theta_{z1}\quad \nu_2\quad \theta_{z2}\quad \nu_3\quad \theta_{z3}\right\}^T$

约束条件 $\nu_1=\theta_{z1}=\nu_3=0$，划去 1、2、5 行列得（注意用上题结果时要以 2l 代 l）

$$\frac{EI}{l}\begin{bmatrix}\frac{36}{l^2} & \frac{16}{l} & \frac{6}{l}\\ \frac{-6}{l} & 12 & 2\\ \frac{6}{l} & 2 & 4\end{bmatrix}\begin{Bmatrix}\nu_2\\ \theta_{z2}\\ \theta_{z3}\end{Bmatrix}=\begin{Bmatrix}P\\ 0\\ 0\end{Bmatrix}$$

图 7.4，由对称计算一半，注意到$\theta_{z2}=0,\nu_3\neq 0$



$$[K]^{(1)}=\frac{EI}{l}\begin{bmatrix}\frac{12}{l^2} & \frac{6}{l} & \frac{-12}{l^2} & \frac{6}{l}\\ \frac{6}{l} & 4 & \frac{-6}{l} & 2\\ \frac{-12}{l^2} & \frac{-6}{l} & \frac{12}{l^2} & \frac{-6}{l}\\ \frac{6}{l} & 2 & \frac{-6}{l} & 4\end{bmatrix}=\begin{bmatrix}K_{11}{}^{(1)} & K_{12}{}^{(1)}\\ K_{21}{}^{(1)} & K_{22}{}^{(1)}\end{bmatrix}$$

$$\xrightarrow{\text{以2l代l,4I代I}}\left[K^{(2)}\right]=\begin{bmatrix}K_{22}{}^{(2)} & K_{23}{}^{(2)}\\ K_{32}{}^{(2)} & K_{33}{}^{(2)}\end{bmatrix}$$

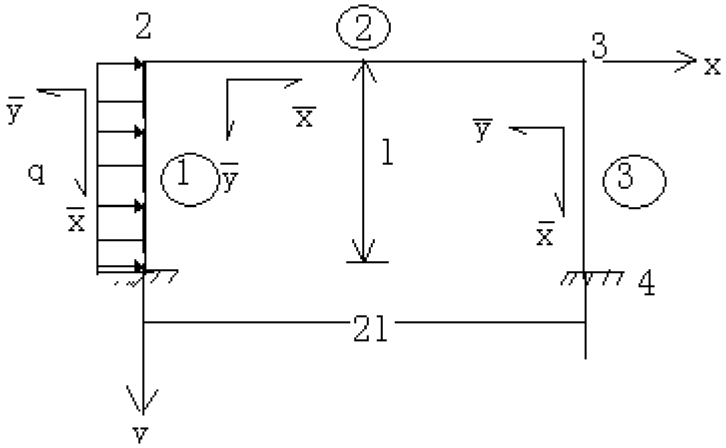
$$\begin{bmatrix}K_{11}{}^{(1)} & K_{12}{}^{(1)} & [0]\\ K_{21}{}^{(1)} & K_{22}{}^{(1)}+K_{22}{}^{(2)} & K_{23}{}^{(2)}\\ [0] & K_{32}{}^{(2)} & K_{33}{}^{(2)}\end{bmatrix}\begin{Bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{Bmatrix}=\begin{Bmatrix}P_1\\ P_2\\ P_3\end{Bmatrix}$$，将各子块代入得

$$\frac{EI}{l}\begin{bmatrix}\frac{12}{l^2} & \frac{6}{l} & \frac{-12}{l^2} & \frac{6}{l} & & \\ \frac{6}{l} & 4 & \frac{-6}{l} & 2 & & \\ \frac{-12}{l^2} & \frac{-6}{l} & \frac{18}{l^2} & 0 & \frac{-6}{l^2} & \frac{6}{l}\\ \frac{6}{l} & 2 & 0 & 12 & \frac{-6}{l} & 4\\ & & \frac{-6}{l^2} & \frac{-6}{l} & \frac{6}{l^2} & \frac{-6}{l}\\ & & \frac{6}{l} & 4 & \frac{-6}{l} & 8\end{bmatrix}\begin{Bmatrix}v_1\\ \theta_{z1}\\ v_2\\ \theta_{z2}\\ v_3\\ \theta_{z3}\end{Bmatrix}=\begin{Bmatrix}R_{y1}+\frac{ql}{2}\\ M_{R1}+\frac{ql^2}{12}\\ -k_2v_2+\frac{3ql}{2}\\ \frac{ql^2}{4}\\ ql\\ M_{R3}\end{Bmatrix}$$

由约束条件 $v_1=\theta_{z1}=\theta_{z3}=0, R_2=-k_2v_2=-\frac{20EI}{l^2}$，划去 1、2、6 行列，将 k_2 代入 $[K]$ 得

$$\frac{EI}{l}\begin{bmatrix}\frac{18+20}{l^2} & 0 & \frac{-6}{l^2}\\ 0 & 12 & \frac{-6}{l}\\ \frac{-6}{l^2} & \frac{-6}{l} & \frac{6}{l^2}\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_{z2}\\ v_3\end{Bmatrix}=\begin{Bmatrix}\frac{3ql}{2}\\ \frac{ql^2}{4}\\ ql\end{Bmatrix}$$

7.3 题



a) 写出各杆元对总体坐标之单元刚度矩阵

$$\left[\overline{K}^{(1)}\right]=\left[\overline{K}^{(3)}\right]=\frac{E}{l}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{12I}{l^2} & \frac{6I}{l} & 0 & \frac{-12I}{l^2} & \frac{6I}{l} \\ 0 & \frac{6I}{l} & 4I & 0 & \frac{-6I}{l} & 2I \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & \frac{12I}{l^2} & \frac{-6I}{l} & 0 & \frac{12I}{l^2} & \frac{-6I}{l} \\ 0 & \frac{6I}{l} & 2I & 0 & \frac{-6I}{l} & 4I \end{bmatrix}$$

$$=\begin{bmatrix} \overline{K}_{22}^{(1)} & \overline{K}_{21}^{(1)} \\ \overline{K}_{12}^{(1)} & \overline{K}_{11}^{(1)} \end{bmatrix}=\begin{bmatrix} \overline{K}_{33}^{(3)} & \overline{K}_{34}^{(3)} \\ \overline{K}_{43}^{(3)} & \overline{K}_{44}^{(3)} \end{bmatrix}\xrightarrow{\text{以2l代l}}\begin{bmatrix} K_{22}^{(2)} & K_{23}^{(2)} \\ K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix}=\left[\overline{K}^{(2)}\right]=\left[K^{(2)}\right]$$

$$[t]=\begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\quad \therefore [T]=\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$$

$$\left[K^{(1)}\right]=\left[K^{(3)}\right]=[T]\left[\overline{K}^{(1)}\right][T]^{-1}$$

$$
=\begin{bmatrix} 0 & -1 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}
\frac{E}{l}\begin{bmatrix} A & 0 & 0 & -A & 0 & 0 \\ 0 & \frac{12I}{l^2} & \frac{6I}{l} & 0 & \frac{-12I}{l^2} & \frac{6I}{l} \\ 0 & \frac{6I}{l} & 4I & 0 & \frac{-6I}{l} & 2I \\ -A & 0 & 0 & A & 0 & 0 \\ 0 & \frac{-12I}{l^2} & \frac{-6I}{l} & 0 & \frac{12I}{l^2} & \frac{-6I}{l} \\ 0 & \frac{6I}{l} & 2I & 0 & \frac{-6I}{l} & 4I \end{bmatrix}
\begin{bmatrix} 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0 \\ & & & -1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}
$$

$$
\frac{E}{l}\begin{bmatrix} \frac{12I}{l^2} & 0 & \frac{-6I}{l} & \frac{-12I}{l^2} & 0 & \frac{-6I}{l} \\ 0 & A & 0 & 0 & -A & 0 \\ \frac{-6I}{l} & 0 & 4I & \frac{6I}{l} & 0 & 2I \\ \frac{-12I}{l^2} & 0 & \frac{6I}{l} & \frac{12I}{l^2} & 0 & \frac{6I}{l} \\ 0 & -A & 0 & 0 & A & 0 \\ \frac{-6I}{l} & 0 & 2I & \frac{6I}{l} & 0 & 4I \end{bmatrix}
=\begin{bmatrix} K_{22}^{(1)} & K_{21}^{(1)} \\ K_{12}^{(1)} & K_{11}^{(1)} \end{bmatrix}
=\begin{bmatrix} K_{33}^{(3)} & K_{34}^{(3)} \\ K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}
$$

b）集成总刚度矩阵

$$
[K]=\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & \\ & K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} & K_{34}^{(3)} \\ & & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}=
$$

$$
\begin{bmatrix}
\frac{12I}{l^2} & 0 & \frac{6I}{l} & -\frac{12I}{l^2} & 0 & \frac{6I}{l} & & & & & & \\
0 & A & 0 & 0 & -A & 0 & & & & & & \\
\frac{6I}{l} & 0 & 4I & -\frac{6I}{l} & 0 & 2I & & & & & & \\
-\frac{12I}{l^2} & 0 & -\frac{6I}{l} & \frac{12I}{l^2}+\frac{A}{2} & 0 & -\frac{6I}{l} & -\frac{A}{2} & 0 & 0 & & & \\
0 & -A & 0 & 0 & \frac{6I}{4l^2}+A & \frac{3I}{2l} & 0 & -\frac{6I}{4l^2} & \frac{3I}{2l} & & & \\
\frac{6I}{l} & 0 & 2I & -\frac{6I}{l} & \frac{3I}{2l} & 6I & 0 & -\frac{3I}{2l} & I & & & \\
& & & -\frac{A}{2} & 0 & 0 & \frac{12I}{l^2}+\frac{A}{2} & 0 & -\frac{6I}{l} & -\frac{12I}{l^2} & 0 & -\frac{6I}{l} \\
& & & 0 & -\frac{6I}{4l^2} & -\frac{3I}{2l} & 0 & \frac{6I}{4l^2}+A & -\frac{3I}{2l} & 0 & -A & 0 \\
& & & 0 & \frac{3I}{2l} & I & -\frac{6I}{l} & -\frac{3I}{2l} & 6I & \frac{6I}{l} & 0 & 2I \\
& & & & & & -\frac{12I}{l^2} & 0 & \frac{6I}{l} & \frac{12I}{l^2} & 0 & \frac{6I}{l} \\
& & & & & & 0 & -A & 0 & 0 & A & 0 \\
& & & & & & -\frac{6I}{l} & 0 & 2I & \frac{6I}{l} & 0 & 4I
\end{bmatrix}
$$

c）写出节点位移及外载荷列阵

$$\{\delta\}^T=\{u_1 \quad v_1 \quad \theta_{z1} \vdots u_2 \quad v_2 \quad \theta_{z2} \vdots u_3 \quad v_3 \quad \theta_{z3} \vdots u_4 \quad v_4 \quad \theta_{z4}\}^T=\{\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4\}^T$$

固端力：

$$\left\{\overline{F}_{局}^{(1)}\right\}^T=\left\{0 \quad \frac{Q}{2} \quad \frac{Ql}{12} \vdots 0 \quad \frac{Q}{2} \quad -\frac{Ql}{12}\right\}^T$$

$$\left\{\overline{F}_{局}^{(2)}\right\}^T=\left\{\overline{F}_{局}^{(3)}\right\}^T=\{0\}$$

$$\left\{\overline{F}_{总}^{(1)}\right\}^T=[\mathrm{T}]\left\{\overline{F}_{局}^{(1)}\right\}=\begin{bmatrix} 0 & -1 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \frac{Q}{2} \\ \frac{Ql}{12} \\ \cdots \\ 0 \\ \frac{Q}{2} \\ \frac{Ql}{12} \end{Bmatrix}=\begin{Bmatrix} -\frac{Q}{2} \\ 0 \\ \frac{Ql}{12} \\ \cdots \\ -\frac{Q}{2} \\ 0 \\ -\frac{Ql}{12} \end{Bmatrix}=\begin{Bmatrix} \overline{F}_2 \\ \cdots \\ \overline{F}_1 \end{Bmatrix}$$

$$\{P\}_{总}=\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}=$$

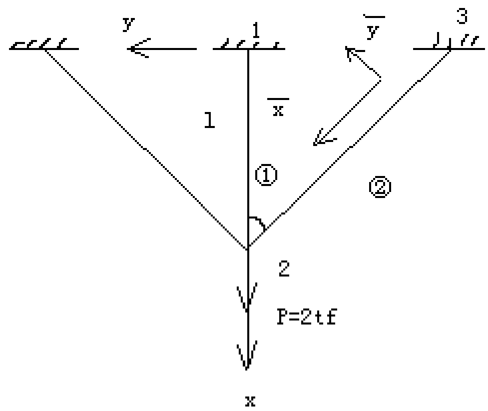
$$\left\{\left(R_{1x}+\frac{Q}{2}\right) \quad T_{y1} \quad \left(M_{R1}+\frac{Ql}{12}\right) \ \vdots\ \frac{Q}{2} \quad 0 \quad -\frac{Ql}{12} \ \vdots\ 0 \quad 0 \quad 0 \ \vdots\ R_{4x} \quad R_{4y} \quad M_{R4}\right\}^T$$

约束处理

$$\begin{bmatrix} K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} \\ K_{32}^{(2)} & K_{33}^{(2)}+K_{33}^{(3)} \end{bmatrix}\begin{Bmatrix} \delta_2 \\ \delta_3 \end{Bmatrix}=\begin{Bmatrix} P_2 \\ P_3 \end{Bmatrix}$$

7.4 题

由对称性，计算图示两个单元即可。



但 $A_{12}=A/2$

P_2 取 P/2 $\left(\hat{x,\overline{x}}\right)=\alpha=45^0$

$$\left[\overline{K}^{(1)}\right]=\left[K^{(1)}\right]=\frac{E(A/2)}{l}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}K_{11}^{(1)}&K_{12}^{(1)}\\K_{21}^{(1)}&K_{22}^{(1)}\end{bmatrix}$$

$$\left[K^{(2)}\right]=[T]\left[\overline{K}^{(2)}\right][T]^{-1}$$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1&0&0\\1&1&0&0\\0&0&1&-1\\0&0&1&1\end{bmatrix}\frac{EA}{\sqrt{2}l}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1&0&0\\-1&1&0&0\\0&0&1&1\\0&0&-1&1\end{bmatrix}$$

$$=\frac{EA}{2\sqrt{2}l}\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}=\begin{bmatrix}K_{33}^{(2)}&K_{32}^{(2)}\\K_{23}^{(2)}&K_{22}^{(2)}\end{bmatrix}$$

$$[K]=\begin{bmatrix}K_{11}^{(1)}&K_{12}^{(1)}&\\K_{21}^{(1)}&K_{12}^{(1)}+K_{22}^{(2)}&K_{11}^{(2)}\\&K_{32}^{(2)}&K_{33}^{(2)}\end{bmatrix}$$

$$=\frac{EA}{2l}\begin{bmatrix}1&0&-1&0&&\\0&0&0&0&&\\-1&0&1+\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\&&-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\&&-\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$$

结构节点位移列阵为

$\{\delta\}=\{u_1,v_1,u_2,v_2,u_3,v_3\}^T$ 其中 $u_1=v_1=u_3=v_3=0, v_2=0$

所以在总刚度矩阵中划去 1，2，4，5，6 组列，设平衡方程为：

$$\begin{Bmatrix}\overline{T}_{x1}^{(1)}\\\overline{T}_{y1}^{(1)}\\\overline{T}_{x2}^{(1)}\\\overline{T}_{y2}^{(1)}\end{Bmatrix}=\frac{EA}{2l}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}0\\0\\u_2\\0\end{Bmatrix}=\frac{EA}{2l}\begin{Bmatrix}-u_2\\0\\u_2\\0\end{Bmatrix}=\frac{P/2}{\left(1+\frac{1}{\sqrt{2}}\right)}\begin{Bmatrix}-1\\0\\1\\0\end{Bmatrix}$$

由于实际 12 杆受力为图示对称情况，

所以 $\overline{T}_{x2}^{(1)}=-\overline{T}_{x1}^{(1)}=\dfrac{P}{\left(1+\frac{1}{\sqrt{2}}\right)}=0.586P=1.172tf$，

对 32 杆

$$\begin{Bmatrix}\overline{U}_2\\\overline{V}_2\end{Bmatrix}=[t]^{-1}\begin{Bmatrix}U_2\\V_2\end{Bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{Bmatrix}U_2\\0\end{Bmatrix}=\frac{1}{\sqrt{2}}\begin{Bmatrix}U_2\\-U_2\end{Bmatrix}$$

$$\begin{Bmatrix}\overline{T}_{x3}\\\overline{T}_{y3}\\\overline{T}_{x2}\\\overline{T}_{y2}\end{Bmatrix}^{(2)}=\frac{EA}{\sqrt{2}l}\begin{bmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}0\\0\\u_2/\sqrt{2}\\-u_2/\sqrt{2}\end{Bmatrix}=\frac{P/2}{\left(1+\frac{1}{\sqrt{2}}\right)}\begin{Bmatrix}-1\\0\\1\\0\end{Bmatrix}$$

所以 23 杆内力为 $\dfrac{P/2}{\left(1+\frac{1}{\sqrt{2}}\right)}=0.586tf$

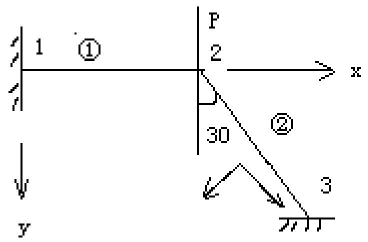
7.5 题

已知：$l_{12}=l_0=200cm, l_{23}=1.155l_0=231cm,$

$I_{12}=I_{23}=I_0=140cm^4,$

$A_{12}=A_{23}=12cm^2, P=6tf, E=2\times10^6\,kg/cm^2$

求：各杆在自 8 坐标系中之杆端力。



解

$$\left[\overline{K}^{(1)}\right]=\frac{E}{l_0}\begin{bmatrix}A&0&0&1-A&0&0\\0&12I/l_0^2&6I/l_0&0&-12I/l_0^2&6I/l_0\\0&6I/l_0&4I&0&-6I/l_0&2I\\-A&0&0&A&0&0\\0&-12I/l_0^2&-6I/l_0&0&12I/l_0^2&-6I/l_0\\0&6I/l_0&2I&0&-6I/l_0&4I\end{bmatrix}=\left[K^{(1)}\right]=\begin{bmatrix}K_{11}^{(1)}&K_{12}^{(1)}\\K_{21}^{(1)}&K_{22}^{(1)}\end{bmatrix}$$

$$\left[\overline{K}^{(1)}\right]=\frac{E}{\beta l_0}\begin{bmatrix}A&0&0&1-A&0&0\\0&12I/(\beta l_0)^2&6I/\beta l_0&0&12I/(\beta l_0)^2&6I/(\beta l_0)^2\\0&6I/\beta l_0&4I&0&-6I/\beta l_0&2I\\-A&0&0&A&0&0\\0&12I/(\beta l_0)^2&-6I/\beta l_0&0&12I/(\beta l_0)^2&-6I/\beta l_0\\0&6I/\beta l_0&2I&0&-6I/\beta l_0&4I\end{bmatrix}=\begin{bmatrix}\overline{K}_{22}^{(2)}&\overline{K}_{23}^{(2)}\\\overline{K}_{32}^{(2)}&\overline{K}_{33}^{(2)}\end{bmatrix}$$

将子快 $\overline{K}_{22}^{(2)}$ 转移到总坐标下 $\left[K_{22}^{(2)}\right]=[t]\left[\overline{K}_{22}^{(2)}\right][t]^T,\left(x\hat{}\,\overline{x}\right)=\alpha=60^0$

$$\left[K_{22}^{(2)}\right]=\frac{E}{2\beta l_0}\begin{bmatrix}1 & -\sqrt{3} & 0\\ \sqrt{3} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A & 0 & 0\\ 0 & 12I/(\beta l_0)^2 & 6I/\beta l_0\\ 0 & 6I/\beta l_0 & 4I\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & \sqrt{3} & 0\\ -\sqrt{3} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

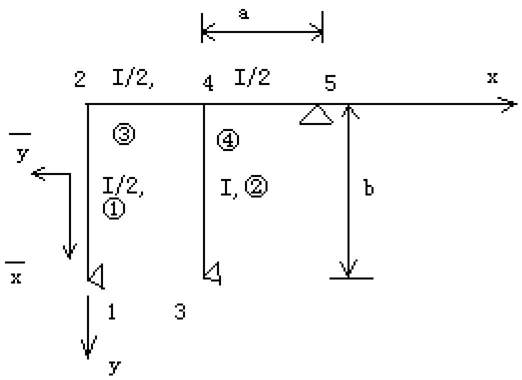
$$=\frac{E}{l_0}\begin{bmatrix}2.618 & 4.487 & -1.363\\ 4.487 & 4.62 & 0.787\\ -1.363 & 0.787 & 121.2\end{bmatrix}$$

约束处理后得：$\left[K_{22}^{(1)}+K_{22}^{(2)}\right]\{\delta_2\}=\{P_2\}$

7.6 题

已知 a=2m, b=1.25a=2.5m, i=4000cm4, I=4i 受均布载荷

a)求$\left[K^{(1)}\right],[K]^{(3)}$ b) $[K]$(用K_{ij}组成)



解：由对称

$$\left[K^{(3)}\right]=\frac{E(i/2)}{a}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 4 & -6/a & 0 & 2 & 6/a\\ 0 & -6/a & 12/a^2 & 0 & -6/a & -12/a^2\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & -6/a & 0 & 4 & 6/a\\ 0 & 6/a & -12/a^2 & 0 & 6/a & 12/a^2\end{bmatrix}=\begin{bmatrix}K_{22}^{(3)} & K_{24}^{(3)}\\ K_{42}^{(3)} & K_{44}^{(3)}\end{bmatrix}$$

$$[K]^{(1)}=\begin{bmatrix}0 & -1 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 0 & -1 & 0\\ & & & 1 & 0 & 0\\ & & & 0 & 0 & 1\end{bmatrix}\frac{E(\frac{I}{2})}{b}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 4 & -\frac{6}{b} & 0 & 2 & \frac{6}{b}\\ 0 & -\frac{6}{b} & \frac{12}{b^2} & 0 & -\frac{6}{b} & -\frac{12}{b^2}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & -\frac{6}{b} & 0 & 4 & \frac{6}{b}\\ 0 & \frac{6}{b} & -\frac{12}{b^2} & 0 & \frac{6}{b} & \frac{12}{b^2}\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0\\ & & & -1 & 0 & 0\\ & & & 0 & 0 & 1\end{bmatrix}$$

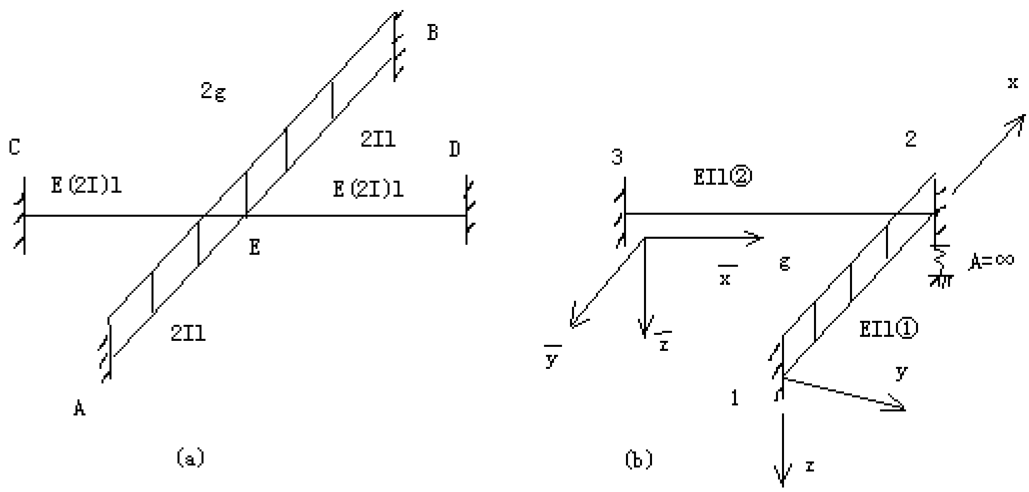
$$
=\frac{2Ei}{b}\begin{bmatrix} 4 & 0 & 6/b & 2 & 0 & -6/b \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6/b & 0 & 12/b^2 & 6/b & 0 & -12/b^2 \\ 2 & 0 & 6/b & 4 & 0 & -6/b \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -6/b & 0 & -12/b^2 & -6/b & 0 & 12/b^2 \end{bmatrix}=\begin{bmatrix} K_{22}^{(1)} & K_{21}^{(1)} \\ K_{12}^{(1)} & K_{11}^{(1)} \end{bmatrix}
$$

$$
[K]=\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & 0 \\ K_{21}^{(1)} & (K_{22}^{(1)}+K_{22}^{(3)}) & 0 & K_{24}^{(3)} & 0 \\ 0 & 0 & K_{33}^{(2)} & K_{34}^{(3)} & 0 \\ 0 & K_{42}^{(3)} & K_{42}^{(2)} & K_{44}^{(3)}+K_{42}^{(2)}+K_{44}^{(4)} & K_{45}^{(4)} \\ 0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(3)} \end{bmatrix}
$$

补充题

用有限元法计算图示平面板架 AB 梁在 E 点剖面的弯矩和弯力，设两梁 AB 及 CD 垂直相交于其中点 E。两梁长度均为 2l，剖面惯性矩均为 2I，弹性模量均为 E，AB 梁能承受的垂直于板架平面的均布荷重为 2g，计算时可不考虑两梁的抗扭刚度。（20 分）

注：可直接应用下式：



（1） 板架中梁元的节点力与节点位移间关系

$$
\begin{Bmatrix} M_{xi} \\ M_{yi} \\ N_{zi} \\ M_{xj} \\ M_{yj} \\ N_{zj} \end{Bmatrix}=\frac{EI}{l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -6/l & 0 & 2 & 6/l \\ 0 & -6/l & 12/l^2 & 0 & -6/l & -12/l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -6/l & 0 & 4 & 6/l \\ 0 & 6/l & -12/l^2 & 0 & 6/l & 12/l^2 \end{bmatrix}\begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ W_i \\ \theta_{xj} \\ \theta_{yj} \\ W_j \end{Bmatrix}
$$

（2） 坐标转换公式：

$$\begin{Bmatrix}\theta_{xi}\\ \theta_{yi}\\ W_i\\ \theta_{xj}\\ \theta_{yj}\\ W_j\end{Bmatrix}=\begin{bmatrix}\cos\alpha & -\sin\alpha & 0 & & & \\ \sin\alpha & \cos\alpha & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & \cos\alpha & -\sin\alpha & 0\\ & & & \sin\alpha & \cos\alpha & 0\\ & & & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\tilde{\theta}_{xi}\\ \tilde{\theta}_{yi}\\ \tilde{W}_i\\ \tilde{\theta}_{xj}\\ \tilde{\theta}_{yj}\\ \tilde{W}_j\end{Bmatrix}$$

[解]

1）由对称性可计算 1/4 板架，取 1，2，3 节点①，②单元，坐标为图 6 有关尺寸，外荷取一半如图示

2）计算单元刚度矩阵

$$\left[\overline{K}^{(1)}\right]=\left[\overline{K}^{(2)}\right]=\frac{EI}{l}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 4 & -6/l & 0 & 2 & 6/l\\ 0 & -6/l & 12/l^2 & 0 & -6/l & -12/l^2\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & -6/l & 0 & 4 & 6/l\\ 0 & 6/l & -12/l^2 & 0 & 6/l & 12/l^2\end{bmatrix}=\begin{bmatrix}K_{33}^{(1)} & K_{12}^{(1)}\\ K_{21}^{(1)} & K_{22}^{(1)}\end{bmatrix}=\begin{bmatrix}\tilde{K}_{33}^{(2)} & \tilde{K}_{12}^{(2)}\\ \tilde{K}_{21}^{(2)} & \tilde{K}_{22}^{(2)}\end{bmatrix}$$

$$\left[K_{22}^{(2)}\right]=[t]\left[\tilde{K}_{22}^{(2)}\right][t]^T=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\frac{EI}{l}\begin{bmatrix}0 & 0 & 0\\ 0 & 4 & 6/l\\ 0 & 6/l & 12/l^2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}=\frac{EI}{l}\begin{bmatrix}4 & 0 & -6/l\\ 0 & 0 & 0\\ -6/l & 0 & 12/l^2\end{bmatrix}$$

$$\left[K_{22}^{(2)}\right]+\left[K_{22}^{(1)}\right]=\frac{EI}{l}\begin{bmatrix}4 & 0 & -6/l\\ 0 & 0 & 0\\ -6/l & 0 & 12/l^2\end{bmatrix}$$

集成总体刚度矩阵：

$$[K]=\begin{bmatrix}K_{11}^{(1)} & K_{12}^{(1)} & \\ K_{21}^{(1)} & K_{22}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)}\\ & K_{32}^{(2)} & K_{33}^{(2)}\end{bmatrix}$$

$$\because \{\delta_1\}=\{\delta_3\}=\{0\}\therefore \left[K_{22}^{(1)}+K_{22}^{(2)}\right]\{\delta_2\}=\{P_2\}$$

即 $$\frac{EI}{l}\begin{bmatrix}4 & 0 & -6/l\\ 0 & 4 & 6/l\\ -6/l & 6/l & 24/l^2\end{bmatrix}\begin{Bmatrix}\theta_{x2}\\ \theta_{y2}\\ W_2\end{Bmatrix}=\begin{Bmatrix}P_{x2}\\ P_{y2}\\ P_{z2}\end{Bmatrix}$$

由约束和对称性：$\theta_{x2}=\theta_{y2}=0$

约束处理：$\frac{EI}{l}\left(\frac{24}{l^2}\right)W_2=\left(P_{z2}\right)=-\left(-ql/2\right)=ql/2\ \therefore W_2=ql^4/48EI$

计算①单元杆端力：

$$\begin{Bmatrix} M_{y1} \\ N_{z1} \\ M_{y2} \\ N_{z2} \end{Bmatrix} = \frac{EI}{l}\begin{bmatrix} 4 & -6/l & 2 & 6/l \\ -6/l & 12/l^2 & -6/l & -12/l^2 \\ 2 & -6/l & 4 & 6/l \\ 6/l & -12/l^2 & 6/l & 12/l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ W_2 \end{Bmatrix} + \begin{Bmatrix} ql^2/l^2 \\ -ql/2 \\ -ql^2/l^2 \\ -ql/2 \end{Bmatrix} = \frac{ql}{2}\begin{Bmatrix} 5l/12 \\ 3/2 \\ -5l/12 \\ -1/2 \end{Bmatrix}$$

实际 AE 杆杆端力为二倍

$$\begin{Bmatrix} M_{EAy} \\ N_{EAy} \end{Bmatrix} = ql\begin{Bmatrix} l/12 \\ -1/2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} M_{EAy} \\ N_{EAy} \end{Bmatrix} = ql\begin{Bmatrix} -l/12 \\ -1/2 \end{Bmatrix}$$

第 9 章 矩形板的弯曲理论

9.1 题（a）已知 a/b=200/60=3.33，q=0.65kg/cm²，k=0（无中面力）

∵a/b>3 且符合荷载弯曲条件 t=1.2cm

$$\sigma_A=\frac{6M_A}{t^2}=\frac{6\cdot qb^2}{t^2\cdot 24}=\frac{0.65\times 60^2}{1.2^2\times 4}=406kg/cm^2$$

$$\sigma_B=\frac{6}{t^2}\cdot qb^2\big/12=\frac{0.65\times 60^2}{1.2^2\times 2}=812kg/cm^2$$

$$\omega_A=gb^4\Big/384E_1\left(t^3\big/12\right)=\frac{(1-\mu^2)}{32}\cdot\frac{gb^4}{Et^3}=0.02843\times\frac{0.65\times 60^4}{2\times 10^6\times 1.2^3}\approx 0.07cm$$

(b) 已知中面力 $\sigma_0=1.88kg/cm^2$

$$\therefore u=\frac{b}{2}\sqrt{\frac{\sigma_0\cdot t\cdot 1}{E_1\cdot t^3/12}}=\frac{b}{2}\sqrt{\frac{12\sigma_0(1-u^2)}{E_1\cdot t^2}}=\frac{60}{2}\sqrt{\frac{12\times 188(1-0.3^2)}{2\times 10^6\times 1.2^3}}=0.8$$

$$\therefore M_A=-\frac{qb^2}{24}\varphi_1(u)=-\frac{0.65\times 60^2}{24}\times 0.925=-90.2kg$$

$$M_B=\frac{qb^2}{12}\chi(u)=\frac{1}{12}\times 0.65\times 60^2\times 0.957=186.6kg$$

$$\omega_A=\frac{gb^4}{384D}f_1(u)=\frac{(1-u^2)}{32}\frac{0.65\times 60^4}{2\times 10^6\times 1.2^3}\times 0.936=0.066cm$$

$$W=1\cdot t^2/6=1\times 1.2^2/6=0.24cm^3$$

$$\therefore \sigma_A=\sigma_0+\left|\frac{M_A}{W}\right|=188+\frac{90.2}{0.24}=563.8kg/cm$$

$$\sigma_A=\sigma_0+\left|\frac{M_B}{W}\right|=188+\frac{186.6}{0.24}=965.5kg/cm$$

与 9（a）比较可见，中面拉力使板弯曲略有改善，如挠度减小，弯曲应力也略有减少，但合成结果应力还是增加了。

9.2 1）当板条梁仅受横荷重时的最大挠度 $\omega_{\max}=\frac{5}{384}\cdot\frac{ql^4}{D}=\frac{5}{384}\frac{5.5\times 80^4(1-0.3^2)}{2\times 10^6\times 2^3/12}$

=0.091<0.2t=0.2×2=0.4 ∴弯曲超静定中面力可不考虑

2）对外加中面力 $\sigma_0=800kg/cm^2$

$$\because u=\frac{l}{2}\sqrt{\frac{12\sigma_0(1-\mu^2)}{Et^2}}=\frac{80}{2}\sqrt{\frac{12\times 800\times 0.91}{2\times 10^6\times 4}}=1.32>0.5$$

∴外加中面力对弯曲要素的影响必须考虑（本题不存在两种中面力复合的情况）

3）

$$\sigma_{A\,\text{下}}^{\ \text{上}} = \sigma_0 \mp \frac{6M_A}{t^2} = 800 \mp \frac{6}{t^2}(\frac{ql^2}{8})\varphi_0(u) = 800 \mp 6/4 \times \frac{0.5\times 80^2}{8} \times 0.58$$

$$= 800 \mp 348 = \begin{Bmatrix} 452 \\ 1148 \end{Bmatrix} kg/cm$$

9.3 已知：t=0.6cm，l=60cm，q=1kg/cm²，$D = \frac{Et^3/12}{1-u^2} = \frac{2\times 10^6 \times 0.6^3}{0.91\times 12} = 39560cm^4$

1）判断刚性：考虑仅受横荷重时的 $\omega_{\max} = \frac{5ql^4}{384}\frac{(1-u^2)}{Et^3/12} = \frac{50\times 60^4 \times 0.91}{384\times 2\times 10^6 \times 0.6^3/12}$

=4.27cm

∴ $\omega_{\max}/t = 4.27/0.6 = 7.1 >> \frac{1}{5}$，必须考虑弯曲中面力。

2）计算超静定中面力（取 k=0.5）

$$\because \sqrt{U} = \frac{1}{\sqrt{K}}\frac{E}{(1-u^2)q}\cdot(\frac{t}{l})^4 = \frac{1}{\sqrt{0.5}}\frac{2\times 10^6}{0.91\times 1}(\frac{0.6}{60})^4 = 0.031$$

∴ $\log 10^4\sqrt{U} = 2.49$ 由图 9-7 查曲线 A 得 U=3.1

由线性查值法：

$$f_0(3.1) = f_0(3) + \frac{f(3.5)-f(3)}{3.5-3}(3.1-3) = 0.213 + (0.166-0.213)\times 0.2 = 0.204$$

$$\varphi_0(3.1) = 0.2 + (0.153-0.2)\times 0.2 = 0.191$$

$$T = D(\frac{2\mu}{L})^2 = 39560\times(\frac{6.2}{60})^2 = 422.4kg$$

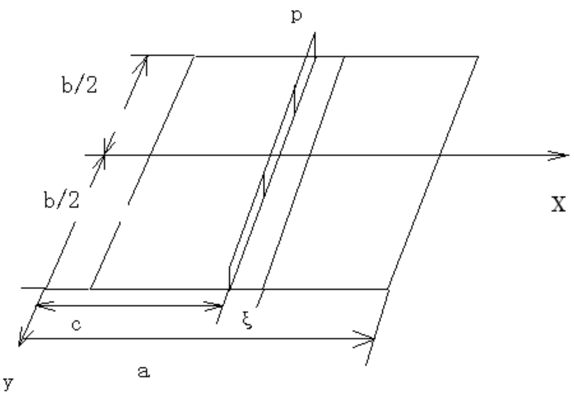
$$\sigma_0 = T/t\cdot 1 = 422.4/0.6 = 704kg/cm^2$$

$$\therefore \omega_{\text{中}} = \frac{5}{384}\frac{ql^4}{D}f_0(u) = \frac{5\times 1\times 60^4 \times 0.204}{384\times 39560} = 0.870$$

$$\sigma_{\max} = \sigma_0 + \frac{6}{t^2}\frac{ql^2}{8}\varphi_0(u) = 704 + \frac{6}{0.6^2}\times\frac{1\times 60^2}{8}\times 0.191 = 2137kg/cm^2$$

9.4 设 $\omega(x,y) = \sum_m f_m(y)\sin\frac{m\pi x}{a}$ 满足 $x=0,a$ 解，代入微方程

$$\frac{\partial^4\omega}{\partial x^4}+2\frac{\partial^4\omega}{\partial x^2\partial y^2}+\frac{\partial^4\omega}{\partial y^4}=\frac{q(x,y)}{D}$$



设关于 $f_m(y)$ 的常微分方程：

$$\sum_m\left[f_m^{IV}(y)-2(\frac{m\pi}{a})^2f_m''(y)+(\frac{m\pi}{a})^4f_m(y)\right]\sin\frac{m\pi x}{a}=\frac{q(x,y)}{D} \qquad (1)$$

为定 $f_m(y)$ 现将 $q(x,y)$ 也展成相应的三角级数：$q(x,y)=\sum_m q_m(y)\sin\frac{m\pi x}{a}$，其中

$$q_m(y)=\frac{2}{a}\int_0^a q(x,y)\sin\frac{m\pi x}{a}dx$$

本题可看成 $q(x,y)=q_0={}^{\eta\cdot b}\!/\!_{\xi\cdot b}={}^{\eta}\!/\!_{\xi}$　（$\xi\to 0$ 的极限情景）

$$\therefore\ q_m(y)=\frac{2}{a}\lim_{\xi\to 0}\int_c^{c+\xi}\frac{\eta}{\xi}\sin\frac{m\pi x}{a}dx=\frac{2\eta}{m\pi}\lim_{\xi\to 0}\frac{\left[\cos\frac{m\pi c}{a}-\cos\frac{m\pi(c+\xi)}{a}\right]}{\xi}$$

$$=\frac{2\eta}{m\pi}\lim_{\xi\to 0}\left[\frac{m\pi}{a}\sin\frac{m\pi(c+\xi)}{a}\right]=\frac{2\eta}{a}\sin\frac{m\pi c}{a}$$

将 $q(x,y)=\sum_m\frac{2\eta}{a}\sin\frac{m\pi c}{a}\sin\frac{m\pi x}{a}$ 代入方程（1）右边比较得

$$f_m^{IV}(y)-2(\frac{m\pi}{a})^2f_m''(y)+(\frac{m\pi}{a})^4f_m(y)=\frac{2\eta}{Da}\sin\frac{m\pi c}{a}$$

特解 $F_m(y)=\frac{2p}{Da}(\frac{a}{m\pi})^4\sin\frac{m\pi c}{a}$ （2）

特征方程：$S^4-2(\frac{m\pi}{a})^2S^2+(\frac{m\pi}{a})^4=0$　$S=\pm(\frac{m\pi}{a})$ 成对双重根

∴齐次解为 $f_m(y)=A_m ch\frac{m\pi}{a}y+B_m sh\frac{m\pi y}{a}+C_m\frac{m\pi}{a}y\cdot ch\frac{m\pi}{a}y+D_m\frac{m\pi}{a}ysh\frac{m\pi}{a}y$

由于挠曲面关于 x 轴对称，所以通解中关于 y 的奇函数必然为 0。（ $B_m = C_m = 0$）

通解：$f_m(y) = A_m ch\frac{m\pi}{a}y + D_m\frac{m\pi}{a}ysh\frac{m\pi}{a}y + F_m(y)$

其中 A_m, D_m 可按 $y = \pm b/2$ 处 $\omega = \partial^2\omega/\partial y^2 = 0$ 即 $f_m(y) = f_m''(y) = 0$ 求解。

即：$$\left.\begin{aligned} &A_m ch\frac{u_m}{2} + D_m\frac{u_m}{2} = -F_m(y) \\ &A_m ch\frac{u_m}{2} + D_m[2ch\frac{u_m}{2} + \frac{u_m}{2}sh\frac{u_m}{2}] = 0 \end{aligned}\right\}\text{式中 } u_m = m\pi b/a$$

解出：$D_m = F_m(y)\Big/2ch\frac{u_m}{2}$ $\qquad A_m = -D_m[2 + \frac{u_m}{2}th\frac{u_m}{2}]$

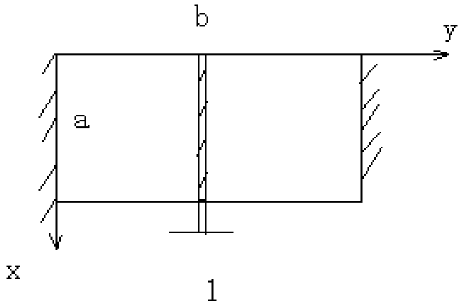
$$\therefore f_m(y) = -\frac{F_m(y)}{2ch\frac{u_m}{2}}\left[2 + \frac{u_m}{2}th\frac{u_m}{2}\right]ch\frac{m\pi}{a}y + \frac{F_m(y)}{2ch\frac{u_m}{2}}\cdot\frac{m\pi}{a}ysh\frac{m\pi}{a}y + F_m(y)$$

$$= \frac{F_m(y)}{2ch\frac{u_m}{2}}\left[\frac{m\pi}{a}ysh\frac{m\pi}{a}y - (2 + \frac{u_m}{2}th\frac{u_m}{2})ch\frac{m\pi}{a}y + 2ch\frac{u_m}{2}\right]$$

$\therefore \omega(x,y) = \sum_m f_m(y)\sin\frac{m\pi x}{a}$ 将（2）中 $F_m(y) = \frac{2\eta a^3}{Dm^4\pi^4}\sin\frac{m\pi c}{a}$ 代入得

$$\omega(x,y) = \frac{pa^3}{D\pi^4}\sum_m \frac{\sin\frac{m\pi c}{a}}{m^4 ch\frac{u_m}{2}}\left[\frac{m\pi}{a}ysh\frac{m\pi}{a}y - (2 + \frac{u_m}{2}th\frac{u_m}{2})ch\frac{m\pi}{a}y + 2ch\frac{u_m}{2}\right]\sin\frac{m\pi x}{a}$$

9.5 已知：a<b　b/a=150/40=3.75, q=0.5kg/cm²



1) 查表得：$k_1 = 0.1356, k_3 = 0.1203, k_4 = 0.1249$

$$\omega_{\max} = k_1 qa^4\big/Et^3 = 0.1356\times 0.5\times 40^4\big/2\times 10^6 = 0.087cm$$

板中心垂直于 x 轴断面应力

$$\sigma_x=(k_3qa^2)\,{}^{6}\!/\!_{t^2}=0.1203\times0.5\times40^2\times6=577kg/cm^2\neq\sigma_{\max}$$

刚固边中点应力：$\overline{\sigma_y}=\sigma_{\max}=k_4qa^2\,{}^{6}\!/\!_{t^2}=0.1249\times0.5\times40^2\times6=600kg/cm^2$

2）按荷形弯曲计算：

$$\omega_{\max}=\omega(a/2,b/2)=\frac{5}{384}qa^4/D=\frac{5}{384}\frac{(1-0.3^2)\times12\times0.5\times40^4}{2\times10^6\times t^3}=0.091cm>0.087$$

板中心垂直于 x 轴断面应力：

$$\sigma_x=\frac{6}{t^2}(\frac{qa^2}{8})=\frac{6\times0.5\times40^2}{1\times8}=600kg/cm^2>577$$

结论：按荷形弯曲计算的结果弯曲要素偏大，所以偏于安全。原因是按荷形弯曲计算时，忽略了短边的影响，按（长边 a）/（短边 b）→∞计算。表中 a/b→∞所对应数值，即表示按荷形弯曲计算结果。

9.6 设 $\omega(x,y)=\sum_m\sum_n\sin\frac{m\pi x}{a}\sin\frac{(2n-1)\pi y}{4b}$ 显然满足几何边界条件

$x=0,a$ 时ω=0，但$\omega'\neq0$

$y=0$ 时ω=0，$\omega'\neq0$

$y=b$时$\omega\neq0$，$\omega'\neq0$

令取一项：$\omega=A\sin\frac{\pi x}{a}\sin\frac{\pi y}{4b}$

则：$\partial\omega/\partial x=A(\frac{\pi}{a})\sin\frac{\pi y}{4b}\cos\frac{\pi x}{a},\partial\omega/\partial y=A(\frac{\pi y}{4b})\sin\frac{\pi x}{a}\cos\frac{\pi y}{4b}$

$\partial^2\omega/\partial x^2=-A(\frac{\pi}{a})^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{4b},\partial^2\omega/\partial y^2=-A(\frac{\pi y}{4b})^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{4b}$

$\partial\omega^2/\partial x\partial y=A(\frac{\pi}{a})(\frac{\pi}{4b})\cos\frac{\pi x}{a}\cos\frac{\pi y}{4b}$

$$\therefore V_{◆}=\frac{D}{2}\int_0^a\int_0^b\left\{\left(\partial^2\omega/\partial x^2+\partial^2\omega/\partial y^2\right)^2+2(1-u)\left[\left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2-\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2}\right]\right\}dxdy$$

$$=\frac{D}{2}\int_0^a\int_0^b\left\{A^2\left[(\frac{\pi}{a})^2+(\frac{\pi}{4b})^2\right]^2\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{4b}+2(1-u)A^2(\frac{\pi}{a})^2(\frac{\pi}{4b})^2\right.$$

$$\left.\left[\cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{4b}-\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{4b}\right]\right\}dxdy$$

$$=\frac{D}{2}\left\{A^2\left[(\frac{\pi}{a})^2+(\frac{\pi}{4b})^2\right]^2\frac{ab}{4}(1-\frac{2}{\pi})+2(1-u)A^2(\frac{\pi^2}{4ab})^2\frac{ab}{4}\left[(1+\frac{2}{\pi})-(1-\frac{2}{\pi})\right]\right\}$$

$$=\frac{A^2\pi^3 D}{2048(ab)^3}\left[(16b^2+a^2)^2(\pi-2)+128(1-u)(ab)^2\right]$$

$$V_{�}=\frac{\mathrm{EI}}{2}\int_0^b\omega_y^{''}(\frac{a}{2},y)dy$$

$$=\frac{\mathrm{EI}}{2}\int_0^b A^2(\frac{\pi}{4b})^4\sin^2\frac{\pi y}{4b}dy$$

$$=\frac{\mathrm{EI}}{2}A^2(\frac{\pi}{4b})^4\cdot\frac{b}{2}(1-\frac{2}{\pi})$$

$$=\frac{\mathrm{EIA}^2\pi^3}{1024\mathrm{b}^3}(\pi-2)$$

$$V=\frac{\mathrm{A}^2\pi^3}{2048(\mathrm{ab})^3}\left\{D\left[\left(16b^2+a^2\right)^2\left(\pi-2\right)+128\left(1-\mu\right)\left(ab\right)^2\right]+2EIa^3\left(\pi-2\right)\right\}$$

$$U=\int_0^a\int_0^b q\omega(x,y)dxdy=q\int_0^a\int_0^b A\sin\frac{\pi x}{a}\sin\frac{\pi y}{4b}dxdy$$

$$=\mathrm{A}\frac{q4ab}{\pi^2}(2-\sqrt{2})$$

$$\frac{\partial(V-U)}{\partial A}=\frac{\mathrm{A}\pi^3}{1024(\mathrm{ab})^3}\left\{D\left[(16b^2+a^2)(\pi-2)+128(1-u)(ab)^2\right]+2EIa^3(\pi-2)\right\}$$

$$-\frac{4qab(2-\sqrt{2})}{\pi^2}$$

$$=0$$

解出：

$$A=\frac{q(ab)^4(4(2-\sqrt{2})/\pi^5)\cdot 1024}{D[(16b^2+a^2)^2(\pi-2)+128(1-u)(ab)^2]+2EIa^3(\pi-2)}$$

$$=\frac{7.84qa^4b^4}{D[(16b^2+a^2)^2(\pi-2)+89.6a^2b^2]+2.28EIa^3}$$

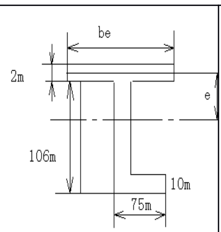
$$\omega(x,y)=\frac{7.84ga^4b^4\sin\frac{\pi x}{a}\sin\frac{\pi y}{4b}}{D[(16b^2+a^2)^2(\pi-2)+89.6a^2b^2]+2.28EIa^3}$$

第 10 章 杆和板的稳定性

10.1 题

（**a**）取板宽 $b_e = \min\left\{\frac{l}{5}, b\right\} = \min\left\{\frac{350}{5}, 75\right\} = 70(\text{cm})$

（但计算 $\lambda = l\big/\sqrt{i/A}$ 中 A 的带板取 75）

	面积 A_i（cm^2）	对参考轴的静矩 $A_iZ_i\,(\text{cm}^3)$	惯性矩 $A_iZ_i^2$ (cm^4)	自身惯性矩 $i_0\,(\text{cm}^4)$
带板	140	0	0	$\frac{1}{12}\times70\times2^3$
立板	10×1	10×（5+1）	10×6^2	$\frac{1}{12}\times1\times10^3$
翌板	6.5×1	6.5×（11-0.5）	6.5×10.5^2	$\frac{1}{12}\times6.5\times1^3$
Σ	156.50	128.25	1076.63	130.54
	A	B	C=1207.17	
	$e = B\big/A = 0.82cm$ $I_{\blacklozenge} = C - Ae^2 = C - B^2\big/A = 1102.1\text{cm}^4$			

$$\lambda = l\Big/\sqrt{I\big/A} = 350\big/\sqrt{1102/166.5} = 136 > 100$$（属大柔度杆）

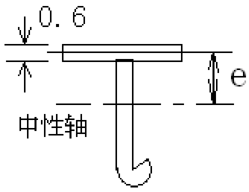
$$\sigma_{cr} = \sigma_E = \pi^2 E/\lambda^2 = \pi^2 2\times10^6/136^2 = 1067$$（kg/cm^2）

（直接由 λ 查图时只能准确到 100kg/cm²，$\therefore \sigma_{cr} = 1100$ kg/cm²）

（**b**）

取代板宽 $b_e = \min\left\{\frac{l}{5}, b\right\} = \frac{l}{5} = 200/5 = 40(\text{cm})$，

求面积 A 时取 $b_e = b = 50$

	面积（cm²）	距参考轴（cm）	静距（cm³）	惯性矩（cm⁴）	自身惯性矩（cm⁴）
带板	40×0.6	0	0		$1/12\times40\times0.6^3$
球扁钢	8.63	6.59	56.87	8.63×6.59^2	85.22
Σ	32.63		56.87	374.78	85.94
	A		B	C=460.72	
	$I=C-B^2/A=361cm^4$				

扶强材两端约束可视为简支

$\lambda = l\Big/\sqrt{I/A} = 200/\sqrt{361/32.63} = 61.13 < 100$（属于小柔度杆）

$\sigma_{cr} = \sigma_y - \sigma_y^2\lambda^2 \big/ 4\pi^2 E = 2400 - 2400^2 \times 65.4^2 \big/ 4\pi^2 \times 2\times10^6 = 2087 kg/cm^2$

（直接查图 F-1 可得 $\sigma_{cr} = 2100 kg/cm^2$）

10.2 题

∵ $\lambda = l/r = 500/5.32 = 94$

查附表曲线得 $\sigma_{cr} = 1800 kg/cm^2$

而实际应力为 P/A

安全系数为 $n = \sigma_{cr}\big/(P/A) = 1800\times42.4\big/30\times10^3 = 2.54$

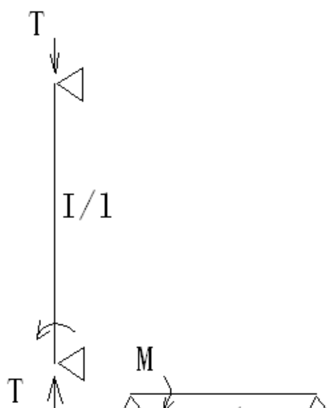
10.3 题

1）写出两杆公共节点的转角连续方程

$\frac{-Ml}{3EI}\psi_1^*(u) = Ml_1\big/3EI_1$

∵M≠0（M=0 表示失稳不属于讨论之列）

∴钢架稳定方程为：



$\psi_1^*(u^*) = -\frac{l_1 I}{l I_1}$ 其中 $u^* = \frac{l}{2}\sqrt{\frac{T}{EI}}$

当 $I_1 = I, l_1 = l$ 时有

$\frac{3}{2u^*}(\frac{1}{2u^*} - \frac{1}{tg2u^*}) = -1$

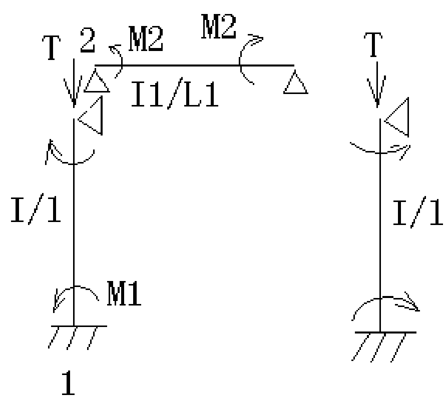
$\psi_1^*(u^*)$	-1.07	-1.04	-1.0039	-1.0011	-.9982	-.995	-.9925
$2u^*$（>π）	3.701	3.710	3.725	3.726	3.727	3.728	3.729

上表用线性内差法求得当 $\psi_1^*(u^*) = -1$ 时，$u^* = 1.863189$ 为最小根

$\therefore T_E = (\frac{2u^*}{l})EI = 3.7263^2\, EI/l^2 = 13.8859\, EI/l^2$

2）如图由对称性考虑 1，2 节点转角方程：

$$\left.\begin{aligned} -\frac{M_1 l}{3EI}\psi_1^*(u^*) - \frac{M_2 l}{6EI}\psi_2^*(u^*) = 0 \\ \frac{M_1 l}{6EI}\psi_2^*(u^*) + \frac{M_2 l}{3EI}\psi_1^*(u^*) = -\frac{M_2 l_1}{3EI} - \frac{M_2 l_1}{6EI} \end{aligned}\right\}$$



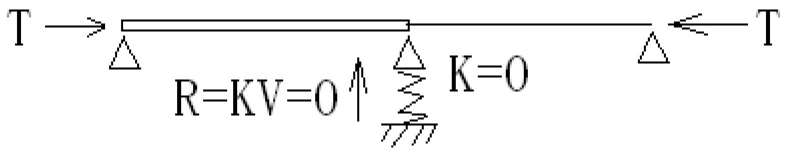
由于失稳时，M_1，M_2 不能同时为 0,这就要求上式方程组关于 M_1，M_2 系数行列式为零，即简化后有稳定方程：

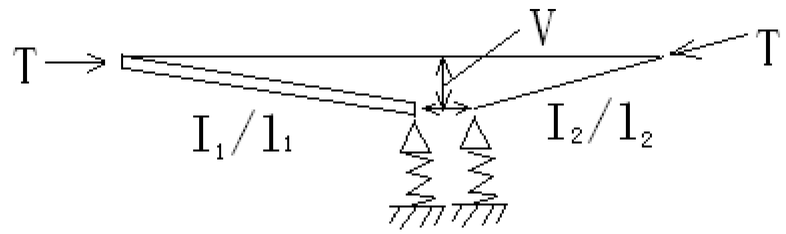
即：$\begin{vmatrix} 2\psi_1^*(u^*) & \psi_2^*(u^*) \\ \psi_2^*(u^*) & 2\psi_1^* + 3\frac{Il_1}{I_1 l} \end{vmatrix} = 0$

$$\psi_2^{*2}(u^*) = 2\psi_1^*(u^*)\left[2\psi_1^* + 3\frac{Il_1}{I_1 l}\right]$$

10.4 题

立截面突变处设弹性支座，列出改点转角连续方程





$$\frac{M(2l_2)}{3E(8I_2)}\psi_1^*(u_1^*)+\frac{v}{2l}=-\frac{Ml_2}{3EI_2}\psi_1^*(u_2^*)-\frac{v}{l_2} \qquad (1)$$

式中：$u_1^*=\frac{l_1}{2}\sqrt{\frac{T}{EI_1}}=\frac{2l_1}{2}\sqrt{\frac{T}{8EI_2}}=\frac{\sqrt{2}}{2}u_2^* \quad \therefore T=(2u_2^*)^2\frac{EI_2}{l_2^2}$

$u_2^*=\frac{l_2}{2}\sqrt{\frac{T}{EI_2}} \quad \therefore 2u_1^*=\sqrt{2}u_2^*$

虚设弹性支座反力 $R=\frac{M+Tv}{(2l_2)}+\frac{M+Tv}{l_2}=0$ （2）

(1) (2)简化关于 M,v 的联立方程组：

$$\left.\begin{aligned} M[\frac{l_2}{12EI_2}\psi_1^*(u_1^*)+\frac{l_2}{3EI_2}\psi_1^*(u_2^*)]+v(\frac{3}{2l_2})=0 \\ M+vT=0 \end{aligned}\right\}$$

失稳时 M，v 不能同时为零，故其系数行列式为零。

即：$\begin{vmatrix} \frac{l_2}{12EI_2}\left[\psi_1^*(u_1^*)+4\psi_1^*(u_2^*)\right] & \frac{3}{2l_2} \\ 1 & \mathrm{T} \end{vmatrix}=0$

化简后稳定方程为：

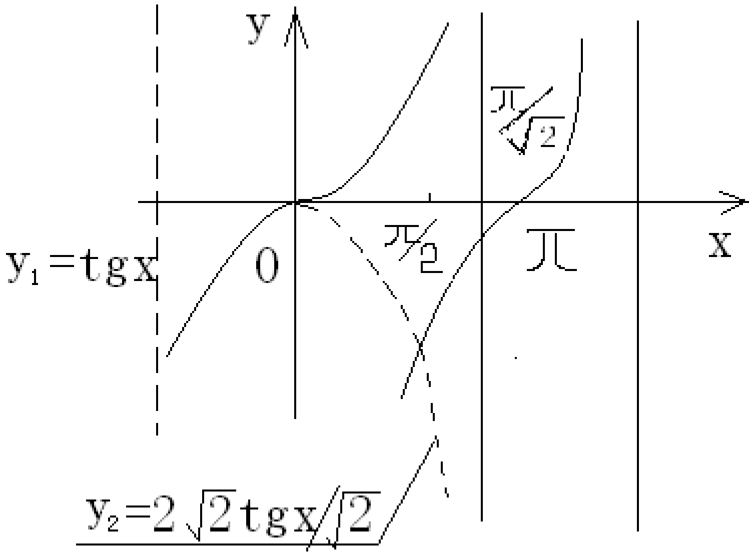
$$2u_2^*\left[\frac{3}{2u_2^*}-\frac{1}{\sqrt{2}tg(\sqrt{2}u_2^*)}-\frac{2}{tg(2u_2^*)}\right]=3 \quad \therefore tgx=-2\sqrt{2}tg(\frac{x}{\sqrt{2}})$$

由图解法或数值解法可得其最小根 $x=(2u_2^*)\min=1.705$ （见下说明）

$$\therefore T_E=\frac{(1.075)^2EI_2}{l_2^2}=2.91\frac{EI_2}{l_2^2}$$

说明：

如下图，最小根 $x=2u_2^*$ 必然在区间（$\frac{\pi}{2},\frac{\pi}{\sqrt{2}}$）内，即（1.57，2.22）



再由数值列表：

x	1.6	1.70	1.705	1.710	1.8
tgx		-7.6966	-7.4065	-7.1372	
$-2\sqrt{2}tg\frac{x}{\sqrt{2}}$		-7.3202	-7.3979	-7.3202	
y_1/y_2		0.9511	1.0012	1.0256	

由线性内差法求解 y_1/y_2 =1 的对应 x 值为：

$$x=2u_2^*=1.70+\frac{1.705-1.70}{1.002-0.9511}(1-0.9511)=1.70488$$

10.5　题

1）计算有关参数：$\nu_1=0,\nu_2=1\Big/\left(1+\frac{2\alpha E_2}{B}\right)=0.5$

$\therefore\ \mu=\mu(\nu_1,\nu_2)\overset{查图}{=}3.36$，n=$\frac{12.5}{2.5}=5$跨

纵骨作为刚支座上连续压杆的欧拉应力

$$\sigma_0=\frac{\pi^2 Ei}{Al^2}=\frac{\pi^2\cdot 2\times10^6\times1250}{64.05\times250^2}=6164kg/cm^2$$

2）求横梁对纵骨的支持刚度：

$$K = \mu^4 EIb \big/ B^4 = 3.36^4 \times 2 \times 10^6 \times 5000 \times 50 \big/ 500^4 = 1019.64 kg/cm$$

横梁临界刚度

$$K_{cr} = \frac{\pi^4 Ei}{l^3} x_j(\lambda)\Big|_{\lambda=1-0} \underset{x_j(1-0)=0.364}{\overset{n=5}{=}} 0.364 \times \frac{\pi^4 \cdot 2 \times 10^6 \times 1250}{250^3} = 5673 kg/cm$$

可见$K < K_{cr} \therefore \sigma_{cr} < \sigma_0$

3）计算弹支座上 5 跨连续压杆的σ_e

$$x_j(\lambda) = I\left(\frac{\mu}{\pi}\right)^4 \left(l \big/ B\right)^3 \frac{b}{B} \cdot \frac{1}{i} = 5000\left(\frac{3.36}{\pi}\right)^4 \left(\frac{2.5}{5}\right)^3 \frac{0.5}{5} \frac{1}{1250} = 0.0654$$

由附图 G-4 查得$\lambda = 0.52$

$$\sigma_E = \lambda \sigma_0 = 0.52 \times 6164 = 3205 kg/cm^2 > \sigma_y = 2400 kg/cm^2$$

需要进行非弹性修正

4）逐步近似法确定σ_{cr}，令$\phi x_j(\lambda) = 0.0654$

由线性内差法计算：

$$\sigma_{cr} = 2050 + \frac{2100-2050}{0.0700-0.0598}(0.0654-0.0598) = 2100 kg/cm^2 < \sigma_y$$

$\sigma_{cr}(kg/cm^2)$	（表 F-1）φ	$\varphi\sigma_0$	$\lambda = \sigma_{cr}/\varphi\sigma_0$	查（$x_j(\lambda)$）	$\varphi x_j(\lambda)$
1600	0.8888	5479	0.2920	0.024	0.0213
1800	0.7500	4623	0.3894	0.040	0.0300
2000	0.5555	3424	0.5841	0.088	0.0488
2050	0.4982	3071	0.6675	0.120	0.05978
2100	0.4375	2697	0.7786	0.160	0.0700

10.6 题

纵式板格尺寸：a=120，b=70，$\therefore a/b = 120/70 = 1.7 > 1 \therefore$稳定系数$K \approx 4$

令$\sigma_{cr} = k\dfrac{\pi^2 D}{b^2 t} = \sigma_y$

即：$\dfrac{k\pi^2}{b} \dfrac{Et^2/12}{(1-u^2)} = \sigma_y$

$$\therefore t^2 = \frac{\sigma_y b^2}{k\pi^2} \frac{(1-u^2)}{E} \cdot 12 = \frac{2400 \times 70^2}{4 \times \pi^2} \frac{0.91}{2 \times 10^6} \times 12 = 1.626$$

$t = 1.28 cm$

10.7 题

已知 l=220cm，t=1.2cm，板 $\sigma_{cr}=\sigma_y=2400kg/cm^2$，求纵骨间距 b

1）$\because \sigma_{cr}=\dfrac{k\pi^2 D}{b^2 t}$　　$\therefore b=\sqrt{\dfrac{k\pi^2 Et^2/12}{\sigma_{cr}(1-u^2)}}=\sqrt{\dfrac{4\pi^2 2\times10^6\times1.2^2}{12\times2400\times0.91}}=65.9cm$

2）要求骨架的临界应力不得小于板的临界应力

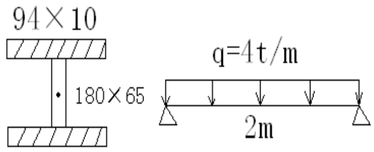
即：$\sigma_{cr}=\dfrac{\pi^2 Ei}{l^2 A}\geq\sigma_E^{板}=\sigma_{cr}=\sigma_y$

式中 i 是纵骨连带板的惯性矩，A=（球扁钢面积）+（带板面积）

解出：$i\geq\dfrac{l^2 A}{\pi^2 E}\sigma_y=\dfrac{220^2\times(11.15+65.9\times1.2)\times2400}{3.14^2\times2\times10^6}=530(cm^4)$

10.8　题

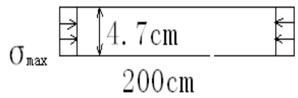
a）组合剖面惯性矩



$I=2\times9.4\times1\times9.5^2+\dfrac{1}{12}\times0.65\times18^3=2012.6cm^4$

$M_{\max}=ql^2/8=40\times200^2/8=2\times10^5(kg\cdot cm)$

$\sigma_{\max}=\mp\dfrac{M_{\max}}{I/h}=\mp\dfrac{2\times10^5}{2012.6/10}=\mp994(kg/cm^2)$



取一半笠板，宽 94/2，长 2m。

设其承受 $\sigma_{\max}=-994kg/cm^2$ 的单向压应力

其边界可视为三边简支，一边完全自由。

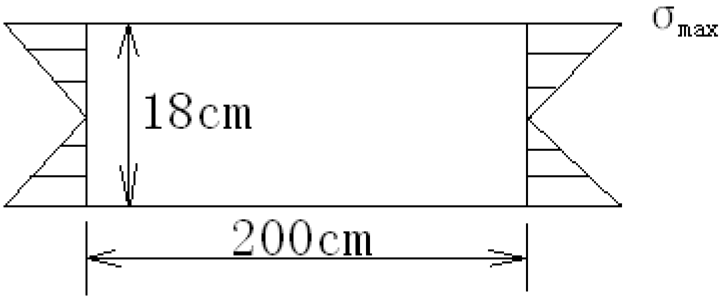
由于长宽比 a/b=200/4.7=42.5〉1

∴稳定系数 k=0.426　　$\therefore\sigma_{cr}=84(100t/b)^2=84(100\times1/4.7)^2=38026kg/cm^2$

∴板的 σ_{cr} 取为 $\sigma_y=2400kg/cm^2$，今 $\sigma_{\max}=994kg/cm^2<\sigma_{cr}=2400kg/cm^2$

故翌板稳定性足够。

b）腹板在纯弯曲正应力（$\eta=2$）作用下计算图形如下



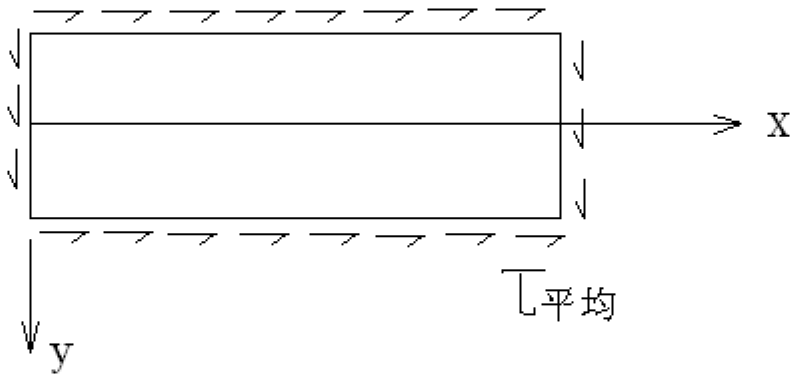
a/b=200/18=11.1〉〉1

取 k=24

$$\sigma_{cr}^{0}=k\frac{\pi^{2}D}{b^{2}t}=\frac{24\times\pi^{2}\times2\times10^{6}\times0.65^{2}}{18^{2}\times0.91\times12}=56571kg/cm^{2}>>\sigma_{y}$$

$$而\sigma_{\max}=\frac{M_{\max}(b/2)}{I}=2\times10^{5}\times9/2012.6=894kg/cm^{2}<\sigma_{y}(安全)$$

c）腹板在建应力作用下稳定计算图形



取 $N=N_{\max}=ql/2=40\times200/2=4000kg$

剪应力沿腹板高度的分布规律为：

$$\tau(y)=\frac{NS_{Z}^{*}}{Ib}=\frac{4000}{2012\times0.65}\left[1\times9.4\times9.5+(9-y)\times0.65\times\frac{(9+y)}{2}\right]$$

$$=354-y^{2}(y=0时\tau_{\max}=354)$$

$$\tau_{平均}=\frac{1}{9}\int_{0}^{9}\tau(y)dy=\frac{1}{9}\int_{0}^{9}(354-y^{2})dy=327kg/cm^{2}$$

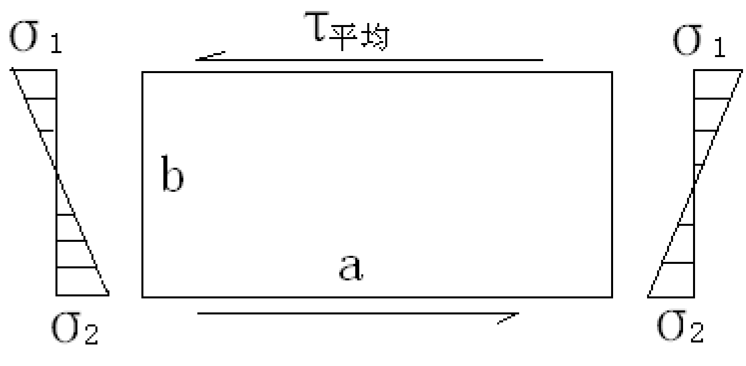
由于腹板的长宽比相当大，故可以近似公式：

$$\tau_{cr}^{0}=1070(\frac{100t}{b})^{2}=1070(\frac{65}{18})^{2}=13952kg/cm^{2}>(\sigma_{y}/2)$$

$$而\tau_{平均}=327kg/cm^{2}<\sigma_{y}/2=1200kg/cm^{2}$$

稳定性足够。

d）腹板在正应力和剪应力共同作用时：



查附录 H-1 No3

计算有关参数：

$$\alpha = a/b = 200/18 = 11.1 > 1$$

$$\beta = \sigma_1/\tau = 894/327 = 2.73$$

$$\chi = \frac{2}{9} + 1/6\alpha^2 = 0.2236$$

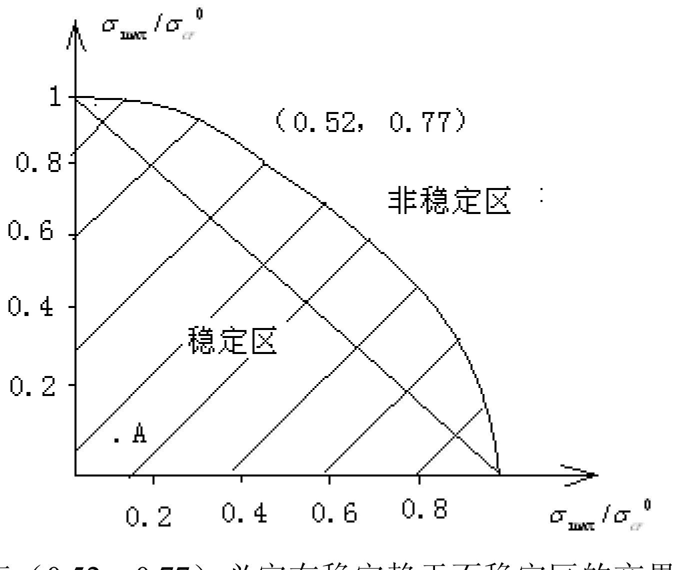
$$k = 24\chi\sqrt{\beta^2+3}\cdot\sqrt{\frac{1}{1+\beta^2\chi^2}} = 24\times 0.2236\sqrt{2.73^2+3}/\sqrt{1+\beta^2\chi^2} = 14.81$$

$$\sigma_i = \frac{\pi^2 E}{12(1-\mu^2)}(t/b)^2 k = \frac{\pi^2\times 2\times 10^6}{12\times 0.91}(0.65/18)^2\times 14.81 = 34910(kg/cm^2)$$

$$\therefore \sigma_{1cr} = \frac{\beta\sigma_i}{\sqrt{\beta^2+3}} = \frac{2.73\times 34910}{\sqrt{2.73^2+3}} = 29478kg/cm^2$$

$$\tau_{cr} = \frac{\sigma_i}{\sqrt{\beta^2+3}} = \frac{\sigma_{1cr}}{\beta} = \frac{29478}{2.73} = 10798kg/cm^2$$

$$\begin{cases}\sigma_{1cr}/\sigma_{cr}^{\ 0} = 29478/56571 = 0.521\\ \tau_{cr}/\tau_{cr}^0 = 10798/13952 = 0.744\end{cases}$$



∴点（0.52，0.77）必定在稳定趋于不稳定区的交界上，过此点 he（0，1）与（1，0）作出凸形边界如图，阴影地区为稳定区。

本题：

$$\begin{cases}\sigma_{\max}/\sigma_{cr}^{\ 0} = 894/56571 = 0.016\\ \tau_{\max}/\tau_{cr}^0 = 354/13952 = 0.025\end{cases}$$

点 A（0.016，0.025）显然落在稳定区内，可见此工字钢腹板在联合受力情况下，其稳定性也是足够的。

10.9 取 $\nu(x) = a_1\sin\frac{\pi x}{l}, \nu'' = -a_1(\frac{\pi}{l})^2\sin\frac{\pi x}{l}, I(x)\begin{cases}I_1 + \frac{I_2 - I_1}{0.2l}(x<0.2l)\\ I_0 \quad (0.2l<x<l/2)\end{cases}$

$$V=2\left[\frac{1}{2}E\int_0^{0.2l}\left[I_1+\frac{I_2-I_1}{0.2l}x\right]\left[-a_1\left(\frac{\pi}{l}\right)^2\sin\frac{\pi x}{l}\right]^2dx+\frac{E}{2}\int_{0.2l}^{0.5l}I_2v''^2dx\right]$$

$$=EI_0\left[\int_0^{0.2l}\left(0.4+\frac{3x}{2}\right)\left(a_1^2\left(\frac{\pi}{l}\right)^4\sin^2\frac{\pi x}{l}\right)dx+\int_{0.2l}^{0.5l}a_1^2\left(\frac{\pi}{l}\right)^4\sin^2\frac{\pi x}{l}dx\right]$$

$$=0.7736EI_0a_1^2\left(\pi/l\right)^3$$

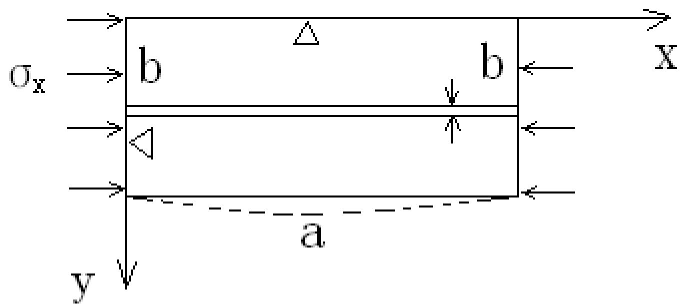
$$U=\frac{1}{2}\int_0^lTv'^2dx=\frac{T}{2}\int_0^la_1^2\left(\frac{\pi}{l}\right)^2\cos^2\frac{\pi x}{l}dx=\frac{T}{4}\frac{\pi^2}{l}a_1^2$$

由$\partial(V-U)/\partial a_1=0$得$1.5472\left(\frac{\pi}{l}\right)^3EI_0a_1=\frac{T\pi^2}{2l}a_1$

由于$a_1\neq0$解出$T_E=9.7213EI_0/l^2$

10.10 题

取$\omega(x,y)=A\frac{y}{b}\sin\frac{\pi x}{a}$



$$\partial\omega/\partial x=A\left(\frac{\pi}{a}\right)\frac{y}{b}\cos\frac{\pi x}{a}$$

$$\partial^2\omega/\partial x^2=-A\left(\frac{\pi}{a}\right)^2\frac{y}{b}\sin\frac{\pi x}{a}$$

$$\partial\omega/\partial y=\frac{A}{b}\sin\frac{\pi x}{a}$$

$$\partial^2\omega/\partial y^2=0$$

$$\partial\omega^2/\partial x\partial y=A\left(\frac{\pi}{a}\right)\frac{1}{b}\cos\frac{\pi x}{a}$$

$$\therefore V=\frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)^2+2(1-\mu)\left[\left(\frac{\partial^2\omega}{\partial x\partial y}\right)^2-\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2}\right]\right\}dxdy$$

$$=\frac{D}{2}\int_0^a\int_0^b\left\{A^2\left(\frac{\pi}{a}\right)^4\frac{y^2}{b^2}\sin^2\frac{\pi x}{a}+2(1-\mu)\left[A^2\left(\frac{\pi}{a}\right)^2\frac{1}{b^2}\cos^2\frac{\pi x}{a}\right]\right\}dxdy$$

$$=\frac{D}{2}\left(\frac{\pi}{ab}\right)^2A^2\int_0^a\int_0^b\left[\left(\frac{\pi}{a}\right)^2y^2\sin^2\frac{\pi x}{a}+2(1-\mu)\cos^2\frac{\pi x}{a}\right]dxdy$$

$$=\frac{D}{2}\left(\frac{\pi}{ab}\right)^2A^2\left[\left(\frac{\pi}{a}\right)^2\frac{ab^3}{6}+(1-\mu)\cdot ab\right]$$

$$=\frac{DA^2\pi^2}{2a}\left[\frac{\pi^2 b}{6a^2}+\frac{(1-\mu)}{b}\right]$$

$$U=\frac{1}{2}\int_0^a\int_0^b\sigma_x t\left(\partial\omega/\partial x\right)^2dxdy$$

$$=\frac{\sigma_x t}{2}\int_0^a\int_0^b A^2\left(\frac{\pi}{ab}\right)^2y^2\cos^2\frac{\pi x}{a}dxdy$$

$$=\frac{1}{12}A^2\pi^2\left(b/a\right)\sigma_x t$$

由 $\partial\Pi/\partial A=0$ 　得 $\frac{DA\pi^2}{a}\left[\frac{\pi^2 b}{6a^2}+\frac{1-\mu}{b}\right]=\frac{A}{6}\pi^2\left(b/a\right)\sigma_x t$

$\because A\neq 0$ 　　$\therefore$解出

$$\sigma_x=\frac{6D}{bt}\left[\frac{\pi^2 b}{6a^2}+\frac{1-\mu}{b}\right]=\frac{\pi^2 D}{b^2 t}\left[\frac{b^2}{a^2}+\frac{6(1-\mu)}{\pi^2}\right]=k\frac{\pi^2 D}{b^2 t}$$

式中$k=\frac{b^2}{a^2}+\frac{6(1-\mu)}{\pi^2}$