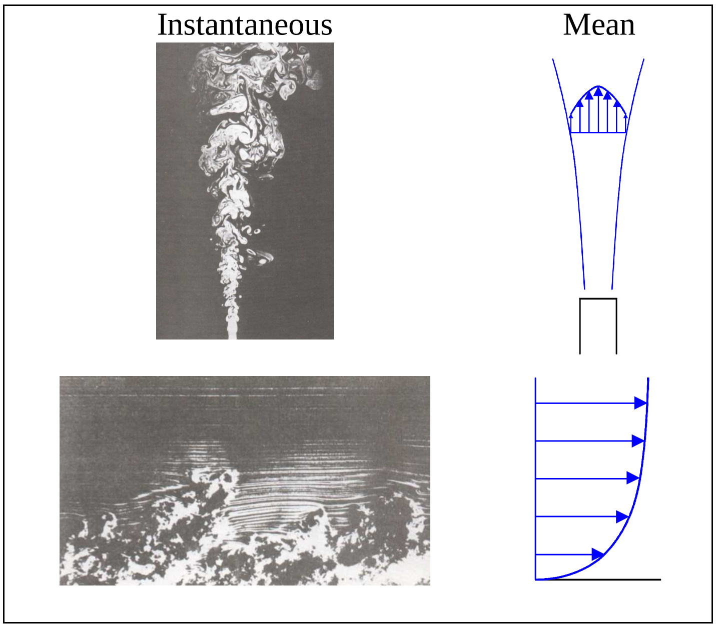
Part (a) - Introduction to turbulence
7.1 What is turbulence?
7.2 Turbulence notation
7.3 Effect of turbulence on the mean flow equations
7.4 Turbulence transport
7.5 Particular shear flows

Part (b) -Turbulence modelling
7.6 Objectives in turbulence modelling
7.7 Eddy-viscosity models
7.8 Advanced turbulence models
7.9 Wall boundary conditions

PART (a) - INTRODUCTION TO TURBULENCE

7.1 What is Turbulence?

- A "random", 3-d, time-dependent eddying motion with many scales, superposed on an often drastically simpler *mean* flow.



- A solution of the Navier-Stokes equations.

- The natural state at high Reynolds numbers (i.e. most civil-engineering flows).

- An efficient mixer ... of mass, momentum, energy, constituents.

- A major source of energy loss.

- "The last great unsolved problem of fluid mechanics".

Laminar, Turbulent, Transitional

Laminar flow is smooth, with adjacent layers of fluid sliding past each other without intermingling. Cross-stream transfer of momentum occurs because viscous forces act between adjacent layers moving at different speeds.

Turbulent flow is "chaotic", with adjacent layers continually distorting and intermingling. A net transport of momentum occurs because of the mixing of fluid elements from different layers with different mean velocity.

Transition from laminar to turbulent flow occurs at sufficiently high Reynolds number:

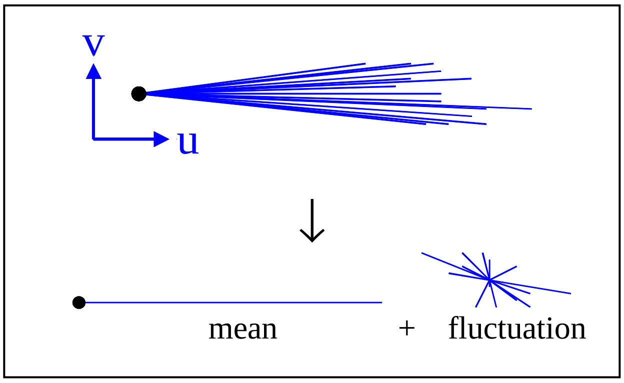
$$\mathrm{Re} \equiv \frac{\rho UL}{\mu} = \frac{UL}{\nu}$$

through a process of local bursts of turbulence coalescing to fill the flow. The value at which transition occurs is called the *critical* Reynolds number, Re_{crit}. It is only approximate, and depends on the smoothness of the surface and the level of disturbances in the flow.

The seminal experiments identifying the role of the Reynolds number in determining the nature of pipe flow were performed by Osborne Reynolds in 1883, in what is now the University of Manchester. The accepted critical Reynolds number for transition in round pipes, based on bulk velocity and pipe diameter, is $\mathrm{Re}_{crit} \approx 2300$.

7.2 Turbulence Notation

The instantaneous value of any flow variable can be decomposed into *mean* + *fluctuation*.



Mean and fluctuating part are denoted by either an overbar (¯) and a prime (′):

$$u = \overline{u} + u'$$

or by upper case and lower case:

$$U + u$$

The first is more commonly used for deriving results but becomes cumbersome in general use. The particular notation being used is, hopefully, obvious from the context.

By definition, the average fluctuation is zero:

$$\overline{u'} = 0$$

The process of taking the mean of a turbulent quantity or product of turbulent quantities is called *Reynolds averaging* and it follows the normal averaging rules for products:

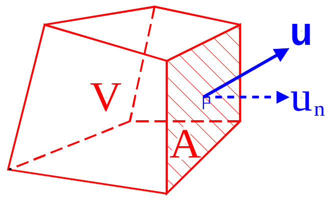
$$\overline{\phi^2} = \bar{\phi}^2 + \overline{\phi'^2} \qquad (\textit{variance})$$

$$\overline{\phi\psi} = \bar{\phi}\bar{\psi} + \overline{\phi'\psi'} \qquad (\textit{covariance})$$

7.3 Effect of Turbulence on the Mean Flow

As a final result engineers usually require only mean quantities. However, in establishing the mean flow, turbulent fluctuations must be taken into account because, although the average value of an individual fluctuation (e.g. $\overline{u'}$ or $\overline{v'}$) is zero, the average value of a *product* (e.g. $\overline{u'v'}$) is usually non-zero and may lead to a significant net flux.

Consider the balance of processes for an arbitrary control volume. For simplicity, assume constant density.



7.3.1 Mass

Instantaneous flow: $\displaystyle\sum_{faces} \rho u_n A = 0$

Average: $$\sum \rho \overline{u_n} A = 0 \qquad (1)$$

The only change is that the instantaneous velocity **u** is replaced by the mean velocity $\bar{\mathbf{u}}$.

The mean velocity satisfies the same mass equation as the instantaneous velocity.

7.3.2 Momentum

Instantaneous flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V u) + \sum_{faces} (\rho u_n A) u = \textit{forces}$$

Average, noting that there is a product of velocities in the momentum flux:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V \bar{u}) + \sum \rho A(\bar{u}_n \bar{u} + \underbrace{\overline{u'_n u'}}_{\textit{extra term}}) = \overline{\textit{forces}}$$

or, rearranging,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V \bar{u}) + \sum (\rho \bar{u}_n A)\bar{u} = \overline{\textit{forces}} + \sum (-\rho\overline{u'_n u'})A \qquad (2)$$

The *mean* momentum equation has exactly the same form as that for the *instantaneous* momentum, except for extra flux terms that have the same effect as an additional stress (force per unit area). These terms,

$$-\rho\overline{u'u'}, \quad -\rho\overline{u'v'}, \quad \text{etc.}, \qquad (3)$$

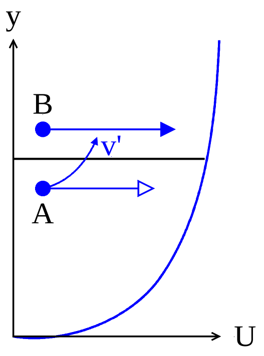
are called the *Reynolds stresses*. These terms arise from the averaging of a *product* of turbulent quantities; they are a consequence of the non-linearity of the fluid-flow equations.

The mean velocity satisfies the same momentum equation as the instantaneous velocity, except for the addition of apparent stresses – the Reynolds stresses.

Thus, a net transfer of momentum may be effected by fluctuating velocities. For example, in a simple shear flow, the net vertical transport of momentum per unit area is the shear stress

$$\tau = \underbrace{\mu \frac{\partial \bar{u}}{\partial y}}_{\text{viscous stress}} \quad \underbrace{-\rho \overline{u'v'}}_{\text{turbulent stress}} \tag{4}$$

Turbulent transport of momentum can be illustrated by considering the motion of particles whose fluctuating velocities allow them to cross a surface drawn in the flow.



If particle A migrates upward ($v' > 0$) then it tends to retain its original momentum, which is now *lower* than its surrounds ($u' < 0$).

If particle B migrates downward ($v' < 0$) it tends to retain its original momentum which is now *higher* than its surrounds ($u' > 0$).

In both cases, $-\rho u'v'$ is positive and, *on average*, tends to reduce the momentum in the upper fluid or increase the momentum in the lower fluid. Hence there is a net transfer of momentum from upper to lower fluid, equivalent to an additional mean stress.

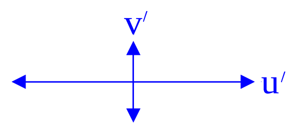
<u>Velocity Fluctuations</u>

Normal stresses: $\overline{u'^2}, \overline{v'^2}, \overline{w'^2}$

Shear stresses: $\overline{v'w'}, \overline{w'u'}, \overline{u'v'}$

Turbulent kinetic energy: $k = \tfrac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$

(Common usage refers to both $-\rho\overline{u'v'}$ and $\overline{u'v'}$ as "turbulent stresses".)



Most turbulent flows are *anisotropic*; i.e. $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$ are different.

7.3.3 General Scalar

In general, the advection of any scalar quantity ϕ gives rise to an additional scalar flux in the mean-flow equations; e.g.

$$\overline{\rho u \phi} = \rho \bar{u} \bar{\phi} \quad + \quad \underbrace{\rho \overline{u'\phi'}}_{\text{additional flux}} \tag{5}$$

Again, the extra term is the result of averaging a product of two fluctuating quantities.

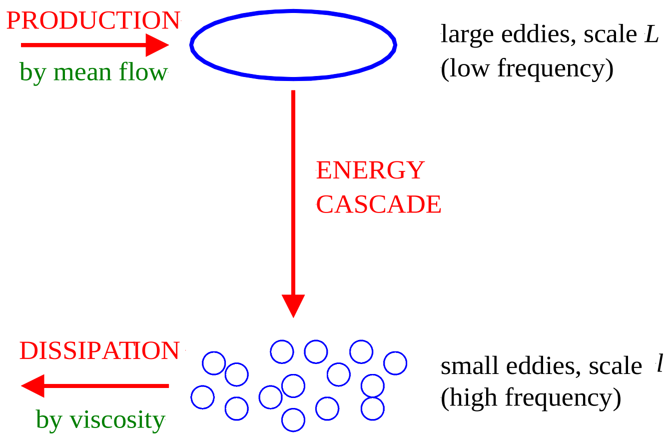
7.3.4 Turbulence Modelling

At high Reynolds numbers, turbulent fluctuations transport a far greater amount of momentum than viscous forces throughout most of the flow. Thus, the modelling of the Reynolds stresses is a vital part of flow prediction.

A *turbulence model* or *turbulence closure* is a means of deriving the Reynolds stresses (and other turbulent fluxes) in order to close the mean-flow equations. Part (b) will describe some of the commoner turbulence models used in engineering.

7.4 Turbulence Transport

7.4.1 Processes



Production and Dissipation

Turbulence is initially generated by instabilities in the flow caused by *mean* velocity gradients (e.g. $\partial U/\partial y$). These eddies in their turn breed new instabilities and hence smaller eddies. The process continues until the eddies become sufficiently small (and *fluctuating* velocity gradients $\partial u/\partial y$ sufficiently large) that viscous effects become important and dissipate turbulence energy as heat. This process – the continual creation of turbulence energy at large scales, transfer of energy to smaller and smaller eddies and the ultimate dissipation of turbulence energy by viscosity – is called the *turbulent energy cascade*.

Transport

It is common experience that turbulence can be *transported* (i.e. carried) with the flow. (Think of the turbulent wake behind a vehicle or downwind of a large building).

7.4.2 Turbulent Transport Equations

It can be proved mathematically (ask me if you are really *desperate* to know how!) that:

(1) Just like the mean momentum components, each Reynolds stress $\overline{u_i u_j}$ satisfies its own scalar transport equation.

(2) Each individual Reynolds stress $\overline{u_i u_j}$ has:

- a *production* term P_{ij} determined by mean velocity gradients;
- a *dissipation* term ε_{ij} formed from viscosity acting on fluctuating velocity gradients;
- a *redistribution* term Φ_{ij} transferring energy between stresses via pressure fluctuations.

These make up the “source” term of the Reynolds-stress transport equation:

$$net\ source = production + redistribution - dissipation$$

There are also “advection” terms (turbulence carried with the flow) and “diffusion” terms (when turbulence intensities vary from point to point).

(3) The production terms for different Reynolds stresses involve different mean velocity gradients; for example, the rate of production per unit mass of $\overline{u_1 u_1} = \overline{u^2}$ and $\overline{u_1 u_2} = \overline{uv}$ are, respectively,

$$\begin{aligned} P_{11} &= -2(\overline{uu}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z}) \\ P_{12} &= -(\overline{uu}\frac{\partial V}{\partial x} + \overline{uv}\frac{\partial V}{\partial y} + \overline{uw}\frac{\partial V}{\partial z}) - (\overline{vu}\frac{\partial U}{\partial x} + \overline{vv}\frac{\partial U}{\partial y} + \overline{vw}\frac{\partial U}{\partial z}) \end{aligned} \tag{6}$$

(*Exercise*: see if you can spot patterns and write down the production terms for the other stresses).

(4) Because:

(i) mean velocity gradients are bigger in some directions than others,

(ii) motions in certain directions are selectively damped (e.g. by buoyancy forces or rigid boundaries),

turbulence is usually *anisotropic*, i.e. $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ are all different.

7.5 Particular Shear Flows

7.5.1 Simple Shear Flows

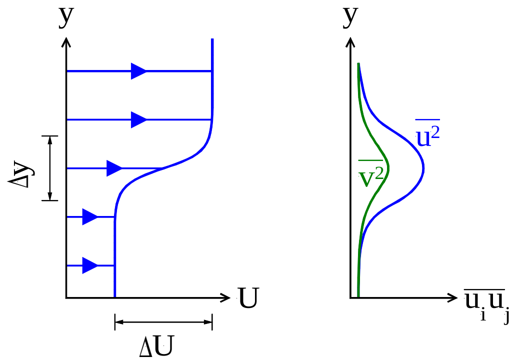
A flow for which there is only one non-zero mean velocity gradient, $\partial U/\partial y$, is called a *simple shear flow*. Because they form a good approximation to many real flows, have been extensively researched in the laboratory and are amenable to basic theory, they are an important starting point for many turbulence models.

For such a flow, the first of (6) and similar expressions show that $P_{11} > 0$ but that $P_{22} = P_{33} = 0$, and hence $\overline{u^2}$ tends to be the largest of the normal stresses because it is the only one with a non-zero production term. On the other hand, the rigid boundary on $y = 0$ selectively damps wall-normal fluctuations; hence $\overline{v^2}$ is the smallest of the normal stresses.

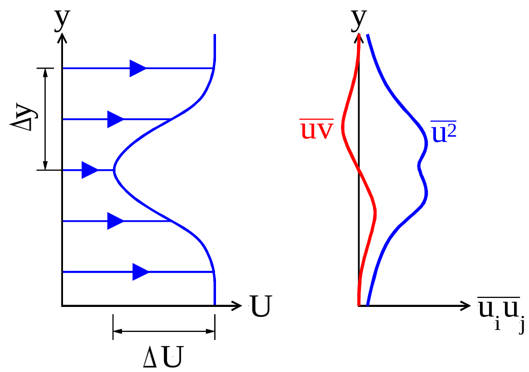
If there are density gradients (for example in atmospheric flows) then buoyancy forces will either damp (stable density gradient) or enhance (unstable density gradient) vertical fluctuations. This represents an interchange of potential energy and kinetic energy.

7.5.2 Free Flows

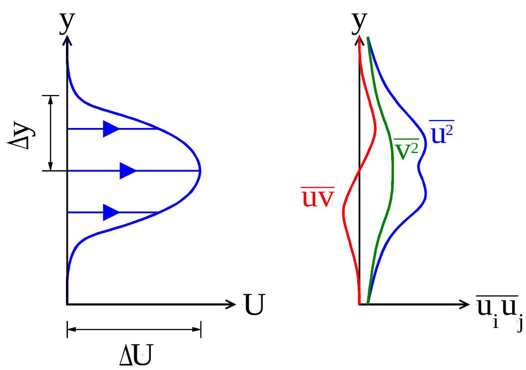
Mixing layer



Wake
(plane or axisymmetric)



Jet
(plane or axisymmetric)



Note that, for these simple flows:

- Maximum turbulence occurs where $|\partial U/\partial y|$ is largest, because this is where *turbulence production* occurs. Note, however, that in the case of wake or jet, some turbulence must have been *diffused* into the central core where $\partial U/\partial y = 0$.

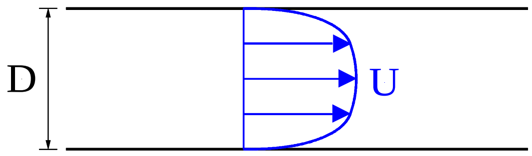
- $\overline{uv}$ has the opposite sign to $\partial U/\partial y$ and vanishes when this derivative vanishes.

- These turbulent flows are anisotropic: $\overline{u^2} > \overline{v^2}$. This is because, for these simple shear flows, turbulence production preferentially favours the streamwise component:

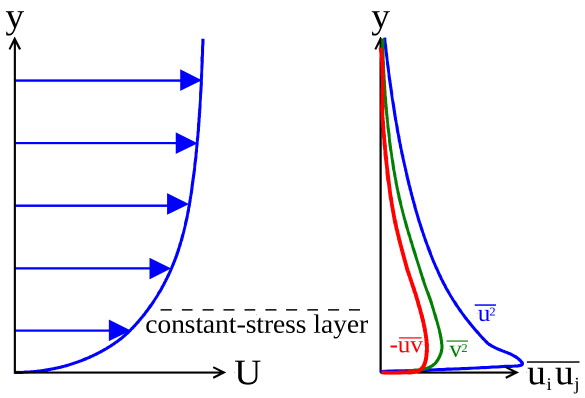
$$P_{11} = -2\overline{uv}\frac{\partial U}{\partial y}, \qquad P_{22} = 0$$

7.5.3 Wall-Bounded Flows

Pipe flow



Flat-plate boundary layer



The presence of a solid boundary leads to flow behaviour and turbulence structure very different from free turbulent flows.

Even though the *bulk* Reynolds number $\mathrm{Re} = U_\infty L/\nu$ is large and hence viscous effects in the majority of the flow are small, there must be a thin layer close to the wall where the *local* Reynolds number based on distance from the wall, $\mathrm{Re}_y = \bar{u}y/\nu$, is small and molecular viscosity is important.

Wall Scales

An important parameter is the *wall shear stress* τ_w (drag per unit area). Like any other stress this has dimensions of [density] × [velocity]^2 and hence it is possible to define a velocity scale called the *friction velocity* u_τ (also written u_*):

$$u_\tau = \sqrt{\tau_w/\rho} \qquad (7)$$

From u_τ and ν it is possible to form a viscous length scale $l_\nu = \nu/u_\tau$, and hence a non-dimensional distance from the wall:

$$y^+ = \frac{yu_\tau}{\nu} \qquad (8)$$

The direct effects of viscosity are only important when y^+ is $O(1)$.

The total mean shear stress is made up of viscous and turbulent parts:

$$\tau = \underbrace{\mu \frac{\partial U}{\partial y}}_{viscous} \underbrace{- \rho \overline{uv}}_{turbulent}$$

When there is no streamwise pressure gradient τ is approximately constant over a significant depth and is equal to the wall stress τ_w. This assumption of *constant shear stress* allows us to establish the velocity profile in regions where either viscous or turbulent stresses dominate.

<u>Near-Wall Region (Viscous Sublayer)</u>

Very close to a smooth wall turbulence is damped out by the presence of the boundary. In this region the shear stress is predominantly viscous:

$$\tau = \mu \frac{\partial U}{\partial y} = \tau_w, \text{ } constant$$

$$\Rightarrow \qquad U = \tau_w \frac{y}{\mu} \tag{9}$$

i.e. *The mean velocity profile in the viscous sublayer is linear*. This is generally a good approximation in the range $y^+ < 5$.

<u>Log-Law Region</u>

At large Reynolds numbers, the turbulent part of the shear stress dominates throughout most of the boundary layer so that, (on largely dimensional grounds, since u_τ and y are the only possible velocity and length scales),

$$\frac{\partial U}{\partial y} \propto \frac{u_\tau}{y}$$

$$\Rightarrow \qquad U = u_\tau \left(\frac{1}{\kappa} \ln \frac{y u_\tau}{\nu} + B\right) \tag{10}$$

i.e. *The turbulent boundary-layer profile is logarithmic*.

κ (*von Kármán's constant*) and B are universal constants with experimentally-determined values of about 0.41 and 5 respectively.

Experimental measurements indicate that the log law actually holds to a good approximation over a substantial proportion of the boundary layer. (This is where the logarithm originates from in common friction-factor formulae such as the Colebrooke-White formula for pipe flow).

(9) and (10) are often written in non-dimensional form:

$$\begin{aligned} &\text{(laminar)} \qquad U^+ = y^+ \\ &\text{(turbulent)} \quad U^+ = \frac{1}{\kappa}(\ln y^+ + B) \end{aligned} \tag{11}$$

where

$$U^+ = \frac{U}{u_\tau} \quad , \qquad y^+ = \frac{y u_\tau}{\nu} \tag{12}$$

Summary of Part (a)

- Turbulence is a 3-d, time-dependent, eddying motion with many scales, causing continuous mixing of fluid.

- Prediction of turbulent flow starts by decomposing each flow variable into a *mean* (or *average*) plus a fluctuation. The process of averaging turbulent variables or their products is called *Reynolds averaging*.

- Turbulent fluctuations make a net contribution to the transport of momentum and other quantities. Turbulence enters the mean momentum equations via the *Reynolds stresses*, e.g.

$$\tau_{turb} = -\rho\overline{u'v'}$$

- A means of specifying the Reynolds stresses (and hence solving the mean flow equations) is called a *turbulence model* or *turbulence closure*.

- Turbulence energy is generated at large scales by mean-velocity gradients (and, sometimes, body forces such as buoyancy). Turbulence is dissipated (as heat) at small scales by viscous action.

- Because of the directional nature of the generating process (i.e. mean-velocity gradients and/or body forces) turbulence is initially anisotropic. Energy is subsequently redistributed amongst the different stress components, primarily by the action of pressure fluctuations.

- Turbulence modelling is, to a large extent, guided by experimental observations and theoretical considerations for simple free flows (mixing layer, jet, wake, grid-generated turbulence) and wall-bounded flows (pipe flow, flat-plate boundary layer).

PART (b): TURBULENCE MODELLING[1]

7.6 Objectives in Turbulence Modelling

The *Reynolds-averaged Navier-Stokes* (RANS) equations are transport equations for the *mean* velocity (U, V, W) and scalars Φ.

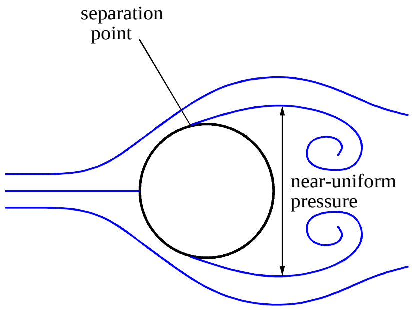
The product of turbulent *fluctuations* contributes to the net transport of momentum via the

Reynolds stresses $-\rho\overline{uv}\,,\ -\rho\overline{u^2}\,,\ \ldots$

and, if there are additional scalars, through the

turbulent fluxes $-\rho\overline{v\phi}$ etc.

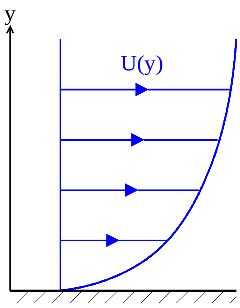
A *turbulence model* is a means of specifying the Reynolds stresses and turbulent fluxes, hence closing the mean-flow equations.



For engineers seeking to predict flow rates, pressure distributions and drag coefficients, the primary requirement of a turbulence model is a good prediction of any dynamically-significant Reynolds stress. Often this is just the shear stress $-\rho\overline{uv}$. If turbulent transport is negligible compared to other forces then a high-quality turbulence model is not required. On the other hand, there are certain flows where an accurate prediction of the shear stress is vital. A particular example is *boundary-layer separation*, because the overall drag is very sensitive to the occurrence and position of separation. High levels of momentum transport enable the main stream to "drag" the near-surface flow forward, countering the adverse pressure gradient and helping to maintain a forward flow, thus delaying separation.

7.7 Eddy-Viscosity Models

7.7.1 The Eddy-Viscosity Hypothesis



In simple shear the mean shear stress is made up of both viscous and turbulent contributions:

$$\tau = \underbrace{\mu\frac{\partial U}{\partial y}}_{viscous} \quad \underbrace{-\rho\overline{uv}}_{turbulent} \tag{13}$$

The single most popular type of turbulence closure is called an *eddy-viscosity model* (EVM) which, by direct analogy with the viscous stress, assumes *stress proportional to rate of strain*:

$$\tau_{turb} \equiv -\rho\overline{uv} \quad = \mu_t\frac{\partial U}{\partial y} \tag{14}$$

μ_t is called an *eddy viscosity* or *turbulent viscosity*.

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[1] Better (but very mathematical) descriptions of turbulence models can be found in:
Wilcox, D.C., 1998, "*Turbulence Modelling for CFD*", 2nd ed, DCW Industries.
Pope, S.B., 2000, "*Turbulent flows*", Cambridge University Press.

The *total* mean shear stress (13) is then

$$\tau = \mu_{eff} \frac{\partial U}{\partial y} \tag{15}$$

where the *effective viscosity* μ_{eff} is the sum of molecular and turbulent viscosities:

$$\mu_{eff} = \mu + \mu_t \tag{16}$$

Notes.

(1) μ is a physical property of the *fluid* and can be measured;
μ_t is a property of the *flow* and must be modelled.

(2) μ_t varies with position.

(3) At high Reynolds-numbers, $\mu_t \gg \mu$ throughout much of the flow.

Eddy-viscosity models are widely used and popular because:

- they are easy to implement in existing solvers (simply use a variable viscosity);
- extra viscosity aids stability;
- they have some theoretical foundation in simple shear flows.

However, one should exercise caution because:

- the eddy-viscosity hypothesis (14) is merely a model; it has little theoretical foundation in complex flows;
- modelling turbulent transport is reduced to a single scalar μ_t and hence at most one Reynolds stress can be represented accurately.

The stress-strain relationship (14) applies only to simple shear flows. To be correct in any reference frame the mathematical representation must be tensorial and given by, for example,

$$\begin{aligned} -\rho\overline{uv} &= \mu_t \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) \\ -\rho\overline{u^2} &= 2\mu_t \frac{\partial U}{\partial x} - \frac{2}{3}\rho k \end{aligned}$$

where k is the turbulent kinetic energy. You should be able to work out expressions for all the other Reynolds stresses by comparison with these.

7.7.2 Reynolds Analogy

The Reynolds analogy assumes a similar *gradient diffusion* model for the turbulent flux of any scalar (e.g., temperature, salt, pollutant):

$$-\rho\overline{v\phi} = \Gamma_t \frac{\partial \Phi}{\partial y} \tag{17}$$

where the *turbulent diffusivity* Γ_t is assumed proportional to the eddy viscosity:

$$\Gamma_t = \frac{\mu_t}{\sigma_t} \tag{18}$$

σ_t is a *turbulent Prandtl number*; its value is usually ≈ 1.

7.7.3 Specifying the Eddy Viscosity

With the eddy-viscosity hypothesis, closure of the mean-flow equations now rests solely on the specification of μ_t, a property of the turbulence.

As with molecular transport ($\nu = \mu/\rho$) it is common to define a *kinematic* eddy viscosity

$$\nu_t = \frac{\mu_t}{\rho} \qquad (19)$$

ν_t has dimensions of [velocity] $\times$ [length], which suggests that it be modelled as

$$\nu_t = u_0 l_0 \qquad (20)$$

On physical grounds, u_0 is a velocity scale reflecting the magnitude of turbulent fluctuations and l_0 a length scale characteristic of the size of turbulent eddies.

For wall-bounded flows a possible candidate for u_0 is the friction velocity $u_\tau = \sqrt{\tau_w/\rho}$. However, a more appropriate velocity scale in general is $\sqrt{k}$, where k is the turbulent kinetic energy.

For wall-bounded flows, l_0 may be proportional to distance from the boundary. For free shear flows (e.g. jet, wake, mixing layer) l_0 may be proportional to the width of the shear layer. However, both of these are geometry-dependent and lack generality. A common practice in general-purpose CFD is to relate l_0 to local turbulence properties (see the k-ε model below).

Common practice nowadays is to solve transport equations for one or more turbulent quantities (usually, k + one other) from which μ_t can then be derived on dimensional grounds. This leads to the following possible classification of eddy-viscosity models based on the number of transport equations to be solved.

zero-equation models:

– constant-eddy-viscosity models;

– mixing-length models: l_0 specified algebraically; u_0 from mean flow gradients.

one-equation models:

l_0 specified algebraically; transport equation to derive u_0;

two-equation models:

transport equations to derive each of u_0 and l_0.

Of these, the most popular in general-purpose CFD are two-equation models: in particular, the k-ε model.

Because they are the most representative types of eddy-viscosity model the mixing-length and k-ε models will be described below.

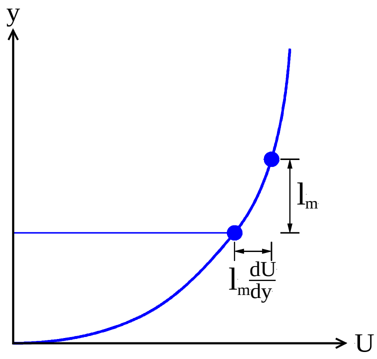
7.7.4 Mixing-Length Models (Prandtl, 1925).

Eddy viscosity:

$$\mu_t = \rho u_0 l_m$$

The *mixing length*, l_m, is a characteristic size of turbulent eddy. If this is specified then the corresponding velocity scale may be deduced (in simple shear) from

$$u_0 = l_m \left| \frac{\partial U}{\partial y} \right| \qquad (21)$$

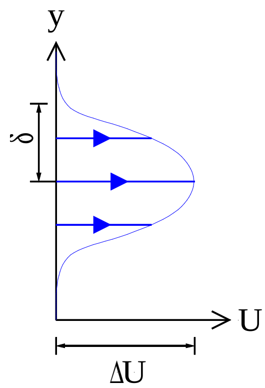


The turbulent shear stress is then

$$\tau = \mu_t \frac{\partial U}{\partial y} \quad = l_m^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y} \qquad (22)$$

The model is based on the premise that if a turbulent eddy displaces a fluid particle by distance l_m, its velocity will differ from its surrounds by an amount $l_m(\partial U/\partial y)$. (Any constant of proportionality can be absorbed into the definition of l_m).

The specification of the *mixing length* l_m depends on the type of flow.

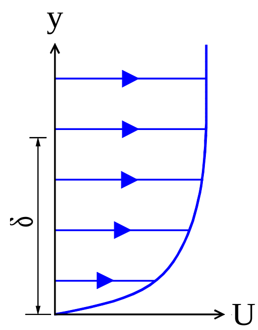


(1) <u>Free shear flows</u>

l_m is assumed constant and proportional to the shear-layer width δ. Rodi (1974) suggests:

$$\frac{l_m}{\delta} = \begin{cases} 0.07 & \text{(mixing layer)} \\ 0.09 & \text{(plane jet)} \\ 0.075 & \text{(round jet)} \\ 0.16 & \text{(plane wake)} \end{cases} \qquad (23)$$

(2) <u>Wall-bounded flows</u>



l_m is typically proportional to the distance from the boundary, y, up to a certain fraction of the boundary-layer height δ. Cebeci and Smith (1974) suggest:

$$l_m = \min(\kappa y, 0.09\delta) \qquad (24)$$

where κ (≈ 0.41) is von Kármán's constant which was introduced earlier in connection with the logarithmic boundary-layer profile.

Mixing-length models work well in near-equilibrium boundary layers, but are difficult to generalise to more complex flows.

7.7.5 The *k*-ε Model

This is probably the most common turbulence model in use today. It is a two-equation eddy-viscosity model with the following specification:

$$\mu_t = C_\mu \rho \frac{k^2}{\varepsilon} \tag{25}$$

C_μ is a constant (with a typical value 0.09) k is the turbulent kinetic energy and ε is the rate of dissipation of turbulent kinetic energy.

k and ε are determined by solving transport equations. For the record (don't learn them!) these are most conveniently given in differential form:

$$\begin{array}{lllll}
\dfrac{\partial}{\partial t}(\rho k) & +\dfrac{\partial}{\partial x_i}(\quad \rho U_i k & -\Gamma^{(k)}\dfrac{\partial k}{\partial x_i} &) = & \rho(P^{(k)} - \varepsilon) \\
\dfrac{\partial}{\partial t}(\rho \varepsilon) & +\dfrac{\partial}{\partial x_i}(\quad \rho U_i \varepsilon & -\Gamma^{(\varepsilon)}\dfrac{\partial \varepsilon}{\partial x_i} &) = & \rho(C_{\varepsilon 1}P^{(k)} - C_{\varepsilon 2}\varepsilon)\dfrac{\varepsilon}{k} \\
\textit{rate of change} & \qquad \textit{advection} & \textit{diffusion} & & \textit{source}
\end{array} \tag{26}$$

(There is an implied summation over the repeated index i – ask me about it if it worries you!).

The diffusivities of k and ε are based on the molecular and turbulent viscosities:

$$\Gamma^{(k)} = \mu + \frac{\mu_t}{\sigma_k}, \qquad \Gamma^{(\varepsilon)} = \mu + \frac{\mu_t}{\sigma_\varepsilon}$$

and, in the standard model (Launder and Spalding, 1974), model constants are:

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1, \quad \sigma_\varepsilon = 1.3 \tag{27}$$

Note that the source term in the k equation is a balance between *production* $P^{(k)}$ and *dissipation* ε. The rate of production (per unit mass) $P^{(k)}$ is given in simple shear by

$$P^{(k)} = \nu_t \left(\frac{\partial U}{\partial y}\right)^2 \tag{28}$$

but the general expression in arbitrary flows is more complex. Under the model assumptions, it is invariably positive and proportional to the square of the mean velocity gradient.

A flow for which $P^{(k)} = \varepsilon$ (production = dissipation) is said to be in *local equilibrium*. A constant-stress, equilibrium boundary layer with logarithmic velocity profile satisfies the high-Reynolds-number (μ negligible) form of (26) provided that the constants satisfy

$$(C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_\varepsilon \sqrt{C_\mu} = \kappa^2 \tag{29}$$

Notes.

(1) The k-ε model is not a single model but a class of slightly different schemes. Many variants include modifications for viscous effects ("low-Reynolds-number k-ε models").

(2) Apart from the diffusion term, the k transport equation is that derived from the Navier-Stokes equation. The ε equation is, however, heavily modelled.

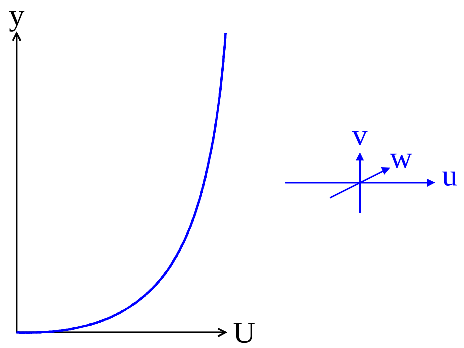
(3) Although k is a logical choice, use of ε as a second scale is not universal and other combinations such as k-ω (ω is a frequency) may be encountered.

7.8 Advanced Turbulence Models

Eddy-viscosity models are popular because:

- they are simple to code;
- extra viscosity aids stability;
- they are supported theoretically in some simple but common types of flow;
- they are very effective in many engineering flows.

However, the dependence of a turbulence model on a single scalar μ_t is clearly untenable when more than one stress component has an effect on the mean flow. The eddy-viscosity model fails to represent turbulence physics, particularly in respect of the different rates of production of the different Reynolds stresses and the anisotropy that results.

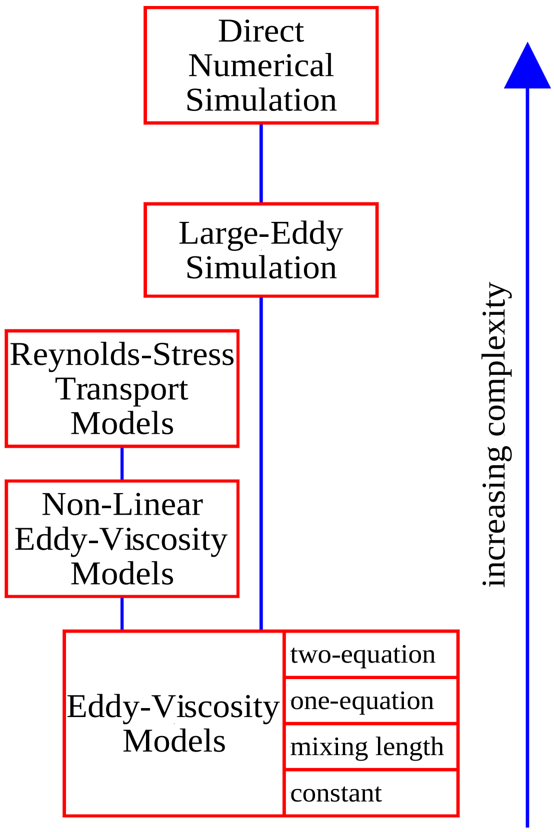


A classic example occurs in simple fully-developed boundary-layer flow where, in the logarithmic region, the various normal stresses are typically in the ratio

$$\overline{u^2} : \overline{v^2} : \overline{w^2} = 1.0 : 0.4 : 0.6 \qquad (30)$$

An eddy-viscosity model would, however, set all of these equal (to $\tfrac{2}{3}k$).

More advanced types of turbulence model (some of which have a proud history at UMIST) are shown right and described below.



7.8.1 Reynolds-Stress Transport Models (RSTM)

Also known as *second-order closure* or *differential stress models*.

Main idea: solve individual transport equations for all stresses, $\overline{u^2}$, $\overline{uv}$ etc., rather than just the turbulent kinetic energy *k*.

These equations are derived from the Navier-Stokes equations. They can be put in the usual canonical form:

rate of change + *advection* + *diffusion* = *source*

but certain terms have to be modelled. The most important balance is in the "source" term, which consists of parts that can be identified as:

– *production* of energy from the mean flow, P_{ij};
– *dissipation* of energy by viscosity, ε_{ij};
– *redistribution* of energy amongst different stress components, Φ_{ij}.

The important point is that, at this level of modelling, both the *advection* term (turbulence carried by the mean flow) and the *production* term (creation of turbulence by the mean flow) are exact. Thus, everything contributing to the energy put into a particular Reynolds-stress component is exact and doesn't need modelling. For example, the rate of production of $\overline{u^2}$ per unit mass is:

$$P_{11} = -2(\overline{u^2}\frac{\partial U}{\partial x} + \overline{uv}\frac{\partial U}{\partial y} + \overline{uw}\frac{\partial U}{\partial z})$$

Assessment.

For:

- Advection and turbulence production terms are *exact* and not modelled; thus, RSTMs should take better account of turbulence physics than eddy-viscosity models.

Against:

- Models are very complex;
- Many important terms (notably redistribution and dissipation) require modelling;
- Models are computationally expensive (6 turbulent transport equations) and tend to be numerically unstable (only the small molecular viscosity contributes to any sort of gradient diffusion).

7.8.2 Non-Linear Eddy-Viscosity Models (NLEVM)[2]

These are a "half-way house" between eddy-viscosity models and Reynolds-stress transport models.

Main idea: extend the simple proportionality between Reynolds-stress and mean-velocity gradients (i.e. rate of strain):

$$\textit{stress} \propto \textit{rate of strain}$$

to a *non-linear* constitutive relation:

$$\textit{stress} = C_1(\textit{rate of strain}) + C_2(\textit{rate of strain})^2 + C_3(\textit{rate of strain})^3 + \ldots$$

(The actual relationship is tensorial and highly mathematical, so has been simplified to words here!)

Models can be constructed so as to reproduce the anisotropy (30) in simple shear flow, as well as a qualitatively-correct response of turbulence in certain other types of flow, notably curved flows. UMIST experience is that a *cubic* stress-strain relationship is desirable.

Assessment.
For:

- Produce qualitatively-correct turbulent behaviour in certain important flows;
- Little more computationally expensive than linear eddy-viscosity models.

Against:

- Doesn't accurately represent production and advection processes;
- Little theoretical foundation in complex flows.

7.8.3 Large-Eddy Simulation (LES)

Simulating a full, time-dependent turbulent flow at large Reynolds number is impractical as it would require huge numbers of control volumes, all smaller than the tiniest scales of motion. *Large-eddy simulation* solves the time-dependent Navier-Stokes equations for the instantaneous (mean + turbulent) velocity that it can resolve on a moderate size of grid and models the subgrid-scale motions that it cannot resolve. The model for the latter is usually very simple, typically a mixing-length model with l_m proportional to the mesh size.

7.8.4 Direct Numerical Simulation (DNS)

This is *not* a turbulence model! It is a solution of the complete time-dependent Navier-Stokes equations without a turbulence model.

This is prohibitively expensive at large Reynolds numbers as huge numbers of grid nodes would be needed to resolve all scales of motion. Nevertheless, supercomputers have extended the Reynolds-number range to a few thousand for simple flows and these results have assisted greatly in the understanding of turbulence physics and development of simpler models.

[2] For details on non-linear eddy-viscosity modelling see, for example:
Apsley, D.D., Chen, W.-L., Leschziner, M.A. and Lien, F.-S., 1997, Non-linear eddy-viscosity modelling of separated flows, *Journal of Hydraulic Research*, **35**, 723-748.

7.9 Wall Boundary Conditions

At walls the no-slip boundary condition applies, so that both mean and fluctuating velocities vanish. At high Reynolds numbers this presents three problems:

- there are very large flow gradients;
- wall-normal fluctuations are selectively damped;
- viscous and turbulent stresses are of comparable magnitude.

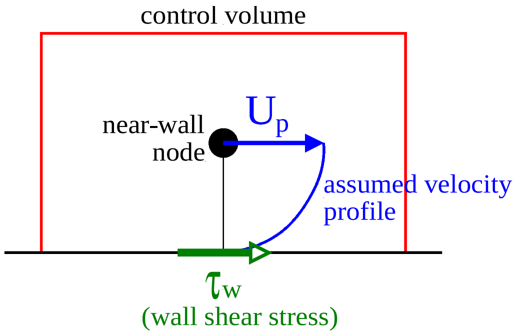
There are two main ways of handling this in turbulent flow.

(1) *Low-Reynolds-number turbulence models*
Resolve the flow right down to the wall. This requires:
– a very large number of nodes;
– special viscosity-dependent modifications to the turbulence model.

(2) *Wall functions*
Don't resolve the near-wall flow completely, but assume theoretical profiles between the near-wall node and the surface.

This doesn't require a large concentration of nodes, but the theoretical profiles used are really only justified in near-equilibrium boundary layers.

7.9.1 Wall Functions



The momentum balance for the near-wall cell requires the wall shear stress τ_w $(= \rho u_\tau^2)$. Because the near-wall region isn't resolved, this requires some assumption about what goes on between near-wall node and the surface.

If the near-wall node lies in the logarithmic region then

$$\frac{U_P}{u_\tau} = \frac{1}{\kappa}\ln(E y_P^+)\,, \qquad y_P^+ = \frac{y_P u_\tau}{\nu} \tag{31}$$

Subscript P denotes the near-wall node. Given U_P and y_P this is solved (iteratively) for u_τ and hence the wall stress τ_w.

If a transport equation is being solved for k a better approach when the turbulence is clearly far from equilibrium (e.g. near separation or reattachment points) is to estimate an "effective" equilibrium friction velocity proportional to $\sqrt{k}$ and estimate the wall shear stress from the tangential velocity U_P and turbulent kinetic energy k_P at the near-wall node:

$$\tau_w = \frac{\kappa u_0 U_P}{\ln(E\dfrac{y_P u_0}{\nu})}\,, \qquad \text{where} \quad u_0 = C_\mu^{1/4} k_P^{1/2} \tag{32}$$

(If the turbulence were genuinely in equilibrium, then u_0 would equal u_τ and (31) and (32) would be equivalent.)

Amendments also have to be made to the turbulence equations, based on assumed profiles for

k and ε. In particular the production of turbulence energy is a cell-averaged quantity, determined by integrating across the cell.

To use these equilibrium profiles effectively, it is desirable that the grid spacing be such that the near-wall node lies within the logarithmic layer, i.e.

$$30 < y_P^+ < 150$$

This means that with wall-function calculations the grid cannot be made arbitrarily fine close to solid boundaries.

Summary of Part (b)

- A turbulence model is a means of specifying the Reynolds stresses (and any other turbulent fluxes), so closing the mean flow equations.

- The most popular closures are eddy-viscosity models, which assume that the Reynolds stress is proportional to the mean strain; e.g. in simple shear:

$$\tau_{turb} \equiv -\rho\overline{uv} \quad = \mu_t \frac{\partial U}{\partial y}$$

- The eddy viscosity μ_t may be specified geometrically (e.g. mixing-length models) or by solving additional transport equations. The most popular combination is the k-ε model (requiring transport equations for turbulent kinetic energy k and its dissipation rate ε).

- More advanced turbulence models include Reynolds-stress transport models, non-linear eddy-viscosity models and large-eddy simulation.

- Wall boundary conditions require special treatment because of large flow gradients and selective damping of wall-normal velocity fluctuations. The main options are low-Reynolds-number models (fine grids) or wall functions (coarse grids).