
3.0 Overview

Fluid dynamics is governed by conservation equations for mass, momentum and energy. The most important of these is the Navier-Stokes equation, which is based upon:

- continuum mechanics;
- the momentum principle;
- a linear stress-strain relationship ($\tau \propto \frac{\partial U}{\partial y}$).

(A fluid for which the last is true is called a *Newtonian* fluid; this is the case for almost all fluids in civil engineering. However, there are some important non-Newtonian fluids – for example, blood, paint and polymer solutions. Specialised CFD techniques exist for these.)

The full equations are time-dependent, 3-dimensional, viscous, compressible, non-linear and highly coupled. However, in most cases it is possible to simplify the analysis by adopting some reduced equation set. Some common approximations are as follows.

Reduction of dimension:
- steady-state;
- 2-d (or axisymmetric).

Neglect of some physical feature:
- incompressible;
- inviscid.

Simplified forces:
- hydrostatic;
- Boussinesq approximation.

Approximations based upon averaging:
- Reynolds-averaging (turbulent flows);
- depth-averaging (shallow-water equations).

Sometimes it may be simpler to solve for *derived* rather than *primitive* variables (ones which can be measured such as ρ, u, p, …); for example:
- potential flow.

3.1 Time-Dependent vs Steady-State

Flows are often assumed steady if the boundary conditions are steady. However, many systems are naturally time-dependent; e.g.

waves;
tides;
pumps;

and many flows with stationary boundaries become time-dependent through instability; e.g.

vortex shedding.

Some solution procedures rely on a time-stepping method to march to steady state; an important example is transonic flow, because the mathematical nature of the governing equations is different for Ma < 1 and Ma > 1.

Consequences. Time-dependent equations are "parabolic"; (1^{st}-order in time; solved by *time marching*). Steady-state equations are "elliptic"; (solved by *implicit, iterative* methods).

3.2 Two- Dimensionality

Geometry and boundary conditions may dictate that the flow is two-dimensional. Two-dimensional calculations require considerably less computer resources.

"Two-dimensional" may be extended to include "axisymmetric". This is actually easier to achieve in the laboratory than true two-dimensionality.

3.3 Incompressibility

A **flow** (not a "fluid", note) is said to be *incompressible* if **flow-induced** pressure and temperature changes do not cause significant density changes. For an incompressible flow, density is constant along a streamline ($\mathrm{D}\rho/\mathrm{D}t = 0$). Note that incompressibility does **not** imply uniform density. Important flows driven by density differences can arise from variations in salinity (oceans) or temperature (atmosphere).

All fluids are compressible to some degree. However, density changes due to flow may be neglected if:

- the Mach number, $\mathrm{Ma} \equiv u/c \ll 1$; (a common rule of thumb is Ma < 0.3);
- absolute temperature changes are small .

These are usually the case in civil engineering; (an important exception is water hammer).

Consequences.
Compressible flow:

- transport equations for density and internal energy (or enthalpy);
- pressure derived from a thermodynamic relation (e.g. the ideal gas law);
- solution by "density-based" methods.

Incompressible flow:

- internal energy is irrelevant: no thermodynamics;
- mass conservation leads to an equation for pressure;
- solution by "pressure-based" methods.

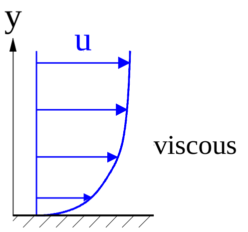
3.4 Inviscid Approximation

If viscosity is neglected, the Navier-Stokes equations become the *Euler equations*.

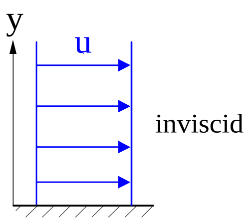
Important consequence. Dropping the viscous term reduces the order of the highest space derivative from 2 to 1. Consider, e.g, streamwise momentum in a developing boundary layer:

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}$$

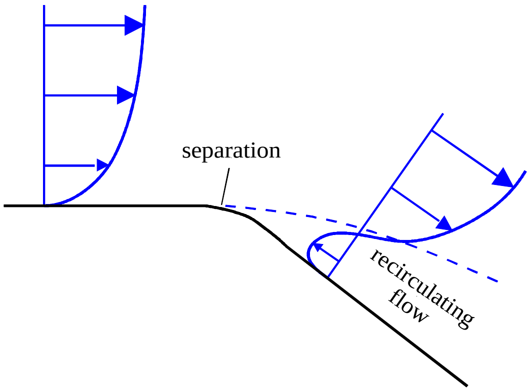
If we drop the viscous term there is no second-order derivative and therefore one less boundary condition.



Very important consequence.
Viscous flows (real fluids) require a non-slip condition (zero velocity) at rigid walls – the *dynamic boundary condition*.

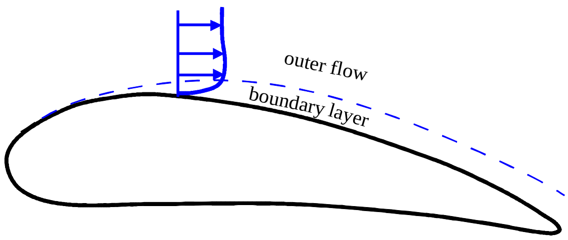


Inviscid (ideal) flows require only the velocity component normal to the wall to be zero – the *kinematic boundary condition*. The wall shear stress is zero.



Although its influence in much of the flow is tiny, the small regions (of large velocity gradients) where viscosity is important can have global effects. For example, it is the viscous boundary layer required to satisfy the non-slip condition that is responsible for *flow separation* in an adverse pressure gradient, whilst for erodible boundaries the surface shear stress is responsible for sediment suspension.

Prandtl's boundary-layer hypothesis: idealise the flow as an outer inviscid layer driven by pressure gradients, matched with a thin inner layer (across which the pressure is effectively constant) to satisfy the no-slip condition. The inner-layer solution is often derived by a forward-marching calculation (see below).



This sort of flow decomposition ("viscid-inviscid interaction") is widely used for aerofoils:

outer flow $\longrightarrow$ pressure distribution and hence "form" lift and drag;
inner flow $\longrightarrow$ viscous drag (small).

The boundary-layer hypothesis is OK if the boundary layer is thin, slowly-developing and doesn't separate. Using a potential-flow method in the outer layer offers considerable computational savings. However, matching-type calculations are seldom used in *general-purpose* codes because:

- they limit the class of flows which can be computed;
- the matching region is difficult to establish *a priori*;
- they don't work when the flow separates.

3.5 Hydrostatic Approximation

The vertical momentum equation can be written as

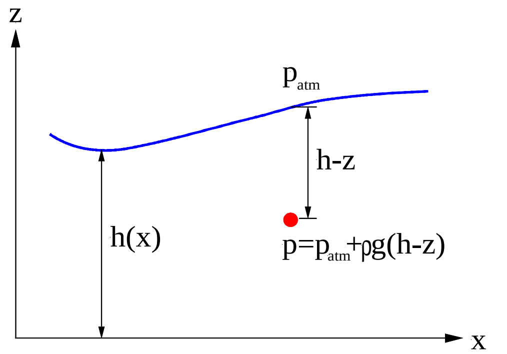
$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g + viscous\ \ forces$$

For large horizontal scales the vertical acceleration D*w*/D*t* is much less than *g* and hence the balance of terms is the same as the hydrostatic law in a quiescent fluid.

$$\frac{\partial p}{\partial z} \approx -\rho g$$

Consequence. With this approximation (in constant-density flows with a free surface) the pressure is determined everywhere by the position of the free surface:

$$p = p_{atm} + \rho g(h - z)\ , \qquad \text{where } h = h(x, y)$$



This results in a huge saving in computational time because the pressure is known everywhere from the surface height and it is unnecessary to solve a separate equation for pressure.

The hydrostatic equation is widely used in conjunction with the depth-averaged shallow-water equations (see below).

3.6 Boussinesq Approximation (Density-Driven Flows)

For *constant-density* flows, pressure and gravitational forces in the *z*-momentum equation can be combined through the *piezometric pressure* $p^* = p + \rho g z$. Gravity need not explicitly enter the momentum equations unless:

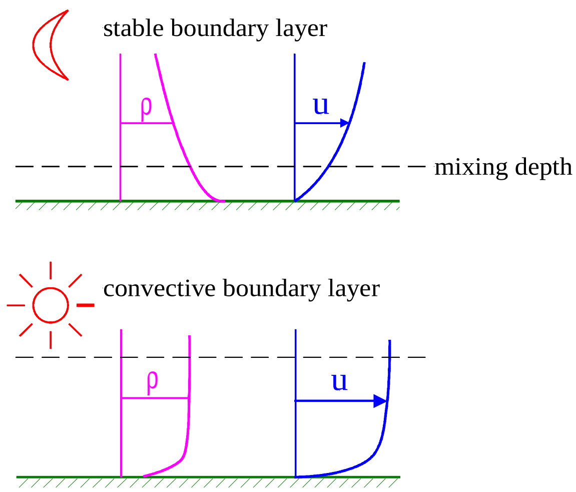
- pressure enters the boundary conditions (e.g. at a free surface);
- there is variable density.

Density variations may arise:

- in compressible flow at high speeds;
- because of changes in temperature or humidity (atmosphere), or salinity (water).

Temperature variations in the atmosphere, brought about by surface (or, occasionally, cloud-top) heating or cooling, are responsible for significant changes in airflow and turbulence:

- on a cold night the atmosphere is *stable* – cool, dense air collects near the surface; vertical motions are suppressed; the boundary layer depth is 100 m or less;
- on a warm day the atmosphere is *unstable* – surface heating produces warm; light air near the surface, convection occurs; the boundary layer may be 2 km deep.



Characteristics of pollution dispersion are very different in the two cases.

If density ρ is a function of some scalar θ (typically, temperature or salinity), then the **relative** change in density is proportional to the change in θ; i.e.

$$\frac{\rho-\rho_0}{\rho_0}=\alpha(\theta-\theta_0)$$

or

$$\rho=\rho_0[1+\alpha(\theta-\theta_0)]$$

where θ_0 and ρ_0 are some reference scalar and density respectively and α is the coefficient of expansion; (the sign convention adopted here is that for salinity, where an increase in salinity lead to a rise in density – the opposite would be true for temperature-driven density changes).

The *Boussinesq approximation* amounts to retaining density variations in the gravitational forces but disregarding them in the advection (mass × acceleration) term; i.e.

$$\rho_0\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p}{\partial z}-\rho g$$

or, in terms of θ,

$$\rho_0\frac{\mathrm{D}w}{\mathrm{D}t}=-\frac{\partial p^*}{\partial z}-\underbrace{\alpha g\rho_0(\theta-\theta_0)}_{buoyancy\ force}$$

The approximation is justified if *relative* density variations are not too large; i.e.

$$\frac{\Delta\rho}{\rho_0}\ll 1$$

This condition is usually satisfied in the atmosphere and oceans.

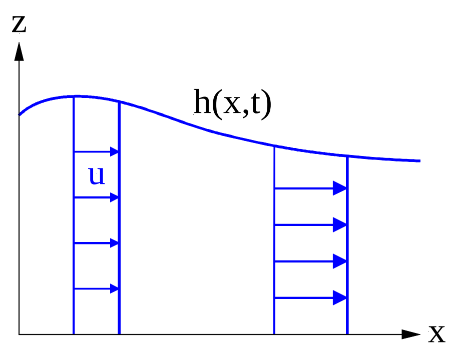
Actually, whilst the Boussinesq approximation is necessary for theoretical work, it is not particularly important in general-purpose CFD, because the momentum and scalar transport equations are solved iteratively and any scalar-driven density variations are easily incorporated at each update.

3.7 Depth-Averaged ("Shallow-Water") Equations

These describe flow of a constant-density fluid with a free surface, where the depth of fluid is small compared with typical horizontal scales.

In this "hydraulic" approximation, the fluid can be regarded as quasi-2d with:
- horizontal velocity components *u*, *v*;
- depth of water, *h*.

Note that *h* may vary due to changes in both free-surface and bed height. The vertical component of velocity may be neglected in comparison with the horizontal velocity.



By applying mass and momentum principles to an arbitrary vertical column of constant-density fluid of height *h*, the depth-integrated equations governing the motion can be written for the one-dimensional (v = 0) case as:

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2 h)}{\partial x} = -\frac{\partial (\frac{1}{2} g h^2)}{\partial x} + \frac{1}{\rho}(\tau_{surface} - \tau_{bed})$$

The $\frac{1}{2} g h^2$ term comes from the total hydrostatic pressure force on a column of water of height h. The net stress (τ) arises from the difference between the surface stress (due to wind) and the bed shear stress (due to friction).

Mathematically, the resulting *shallow-water equations* are analogous to those for a compressible gas. There are direct analogies between hydraulic jumps (shallow-water flow) and shocks (compressible flow). In both cases there is a characteristic wave speed ($\sqrt{gh}$ in the hydraulic case, $\sqrt{\gamma p/\rho}$ in compressible flow). Depending on whether this is greater or smaller than the flow velocity determines whether disturbances can propagate upstream. In the hydraulic case this is determined by the *Froude number*

$$\mathrm{Fr} = \frac{u}{\sqrt{gh}}$$

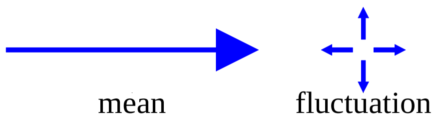
whilst in the high-speed-flow case it is the *Mach number*

$$\mathrm{Ma} = \frac{u}{c}$$

3.8 Reynolds Averaged Equations (Turbulent Flow)

The majority of flows encountered in engineering are turbulent. Most, however, can be regarded as small time-dependent and 3-d *fluctuations* superimposed on a much simpler *mean* flow. Generally, we are only interested in the mean quantities – the mean flow itself or *root-mean-square* (*rms*) levels of turbulence – rather than details of the time-dependent flow.

The process of *Reynolds-averaging* (named after Osborne Reynolds, first Professor of Engineering at Owens College, later to become the Victoria University of Manchester) decomposes each flow variable into mean and turbulent parts:



$$u = \underset{mean}{\overline{u}} + \underset{fluctuation}{u'}$$

The "mean" may be a *time average* (this is usually what is measured in the laboratory) or an *ensemble average* (a hypothetical statistical average over a large number of identical experiments).

When the averaging process is applied to the Navier-Stokes equation, the result is:

- an equivalent equation for the *mean* flow,

  except for

- turbulent fluxes, $-\rho\overline{u'v'}$ etc. (called the *Reynolds stresses*) which provide a net transport of momentum.

For example, the viscous shear stress

$$\tau_{visc} = \mu \frac{\partial \overline{u}}{\partial y}$$

is supplemented by an additional turbulent stress (see Section 7):

$$\tau_{turb} = -\rho \overline{u'v'}$$

In order to solve the mean-flow equations, a *turbulence model* is required for these turbulent stresses. Popular models exploit an analogy between viscous and turbulent transport and employ an *eddy viscosity* μ_t to supplement the molecular viscosity. Thus,

$$\tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u'v'} \quad \to \quad (\mu + \mu_t) \frac{\partial \overline{u}}{\partial y}$$

This is readily incorporated into the mean momentum equation because it simply requires a (position-dependent) *effective viscosity*. However, actually specifying μ_t is by no means trivial – see the lectures on turbulence modelling.

3.9 Potential Flow

In constant-density flows the angular momentum of a fluid element can only change due to the action of viscous forces (since pressure forces always act perpendicular to a surface and cannot impart rotation). For an ideal (inviscid) fluid the flow can be regarded as *irrotational*. For such flows it can be shown that the velocity **u** can be related to a *velocity potential* via:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

Substituting in the continuity equation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, produces a Laplace equation for ϕ:

$$\nabla^2 \phi = 0$$

Important consequence. The entire 3-d flow field is completely described by a single scalar equation.

Velocity components u, v and w are obtained by differentiating ϕ. Pressure is then recoverable from Bernoulli's theorem:

$$p + \tfrac{1}{2}\rho U^2 = \textit{constant} \text{ (along a streamline)}$$

This often gives an adequate description of the flow and pressure fields for real fluids, except very close to solid surfaces where viscous forces are significant. Since Laplace's equation occurs in many branches of physics, a lot of good solvers exist. However, in ignoring the effects of viscosity it says absolutely nothing about the tangential stresses on boundaries and leads, in particular, to the erroneous conclusion that an object moving through a fluid experiences no drag.