

- 2.1 General control-volume form of the governing equations
- 2.2 Fluid-flow equations
 - 2.2.1 Mass (continuity)
 - 2.2.2 Momentum
 - 2.2.3 General scalar
 - 2.2.4 Momentum components as transported scalars
 - 2.2.5 Non-gradient diffusion
 - 2.2.6 Moving control volumes
- 2.3 Other differential equations for fluid flow
 - 2.3.1 Material derivatives and non-conservative form of the flow equations
 - 2.3.2 Equations for derived variables
- 2.4 Compressible and incompressible flow
- 2.5 Non-dimensionalisation
- 2.6 Summary

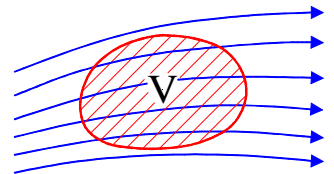
Fluid dynamics is governed by conservation of:

- mass;
- momentum;
- energy;
- any additional constituents.

These basic physical principles can be expressed mathematically in many different ways. Because they form the basis of the finite-volume method for CFD we will focus on **control-volume** (i.e. **integral**) forms of the governing equations, but show also how these are related to equivalent **differential** equations which are easier to write down and manipulate.

2.1 General Control-Volume Form of the Governing Equations

The rate of change of some quantity within an arbitrary control volume is determined by the net rate of transport across its bounding surface (“*flux*”) and the rate of production within that control volume (“*source*”):



$$\left(\begin{array}{c} \text{RATE OF CHANGE} \\ \text{inside } V \end{array} \right) + \left(\begin{array}{c} \text{FLUX} \\ \text{out of boundary} \end{array} \right) = \left(\begin{array}{c} \text{SOURCE} \\ \text{inside } V \end{array} \right) \quad (1)$$

Flux (= rate of transport across a surface) can be divided into:

- *advection* (others prefer *convection*): movement with the flow;
- *diffusion*: net transfer across a surface due to molecular or turbulent fluctuations.

$$\left(\begin{array}{c} \text{RATE OF CHANGE} \\ \text{inside } V \end{array} \right) + \left(\begin{array}{c} \text{ADVECTION + DIFFUSION} \\ \text{through boundary} \end{array} \right) = \left(\begin{array}{c} \text{SOURCE} \\ \text{inside } V \end{array} \right) \quad (2)$$

The finite-volume method is a natural discretisation of this.

2.2 Fluid-flow equations

2.2.1 Mass (Continuity)

Physical principle (mass conservation): mass is neither created nor destroyed.

For any control volume:

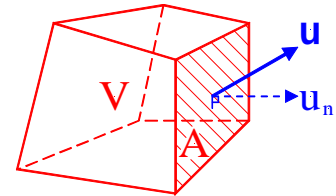
$$\frac{d}{dt}(\text{mass}) + \text{net outward mass flux} = 0 \quad (3)$$

For an arbitrary control volume (or “cell”) with volume V :

mass of fluid: ρV

For a typical cell face with area A and component of velocity along the (outward) normal u_n :

mass flux through cell face: $C = \rho u_n A$



Hence:

$$\frac{d}{dt}(\text{mass}) + \text{net outward mass flux} = 0$$

$$\frac{d}{dt}(\rho V) + \sum_{\text{faces}} \rho u_n A = 0 \quad (4)$$

Special case – steady, 1D flow:

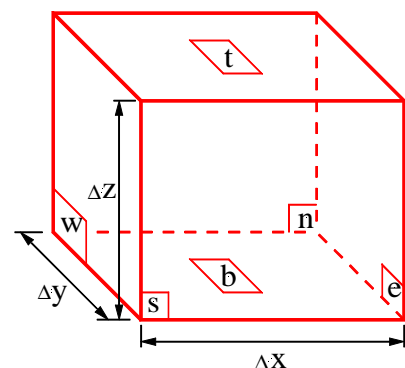
$$\rho_2 u_2 A_2 - \rho_1 u_1 A_1 = 0$$

$$(\text{mass flux})_{\text{out}} - (\text{mass flux})_{\text{in}} = 0$$



A corresponding *conservative differential form* of the mass-conservation equation can be derived by considering a **fixed** control volume with sides Δx , Δy and Δz as shown.

The volume is $\Delta x \Delta y \Delta z$. The “west” and “east” face areas are $A_w = A_e = \Delta y \Delta z$ etc. Variables are assumed approximately constant over cell faces.



Using (4),

$$\underbrace{\frac{d(\rho V)}{dt}}_{\text{rate of change of mass}} + \underbrace{(\rho u A)_e - (\rho u A)_w + (\rho v A)_n - (\rho v A)_s + (\rho w A)_t - (\rho w A)_b}_{\text{net mass flux OUT of control volume}} = 0$$

Noting that $V = \Delta x \Delta y \Delta z$ and $A_w = A_e = \Delta y \Delta z$ etc.,

$$\frac{d(\rho \Delta x \Delta y \Delta z)}{dt} + [(\rho u)_e - (\rho u)_w] \Delta y \Delta z + [(\rho v)_n - (\rho v)_s] \Delta z \Delta x + [(\rho w)_t - (\rho w)_b] \Delta x \Delta y = 0$$

Dividing by the volume, $\Delta x \Delta y \Delta z$:

$$\frac{\partial \rho}{\partial t} + \frac{(\rho u)_e - (\rho u)_w}{\Delta x} + \frac{(\rho v)_n - (\rho v)_s}{\Delta y} + \frac{(\rho w)_t - (\rho w)_b}{\Delta z} = 0$$

Proceeding to the limit $\Delta x, \Delta y, \Delta z \rightarrow 0$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (5)$$

This analysis is directly analogous to the finite-volume procedure, except that in the latter case the control volume does not shrink to zero: **finite-volume** not **infinitesimal volume**.

(**** MSc course only ****)

More mathematically, for an arbitrary volume V with surface ∂V :

$$\frac{d}{dt} \int_V \rho \, dV + \oint_{\partial V} \rho \mathbf{u} \cdot d\mathbf{A} = 0 \quad (6)$$

For a fixed volume, take d/dt under the integral sign and apply the divergence theorem to the surface integral:

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right\} dV = 0$$

Since V is arbitrary, the integrand must be identically zero. Hence,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (7)$$

2.2.2 Momentum

Physical principle (Newton's second law): rate of change of momentum = force

The total rate of change of momentum is made up of:

- the rate of change of **momentum inside the control volume**;
- the difference between **momentum flux leaving and entering**; (i.e. **net flux**).

Hence,

$$\underbrace{\frac{d}{dt}(\text{momentum}) + \text{net outward momentum flux}}_{\text{total rate of change of momentum}} = \text{force} \quad (8)$$

For a cell with volume V and a typical face with area A :

$$\text{momentum in cell} = \text{mass} \times \text{velocity} = \rho V \mathbf{u}$$

$$\text{momentum flux} = \text{mass flux} \times \text{velocity} = (\rho u_n A) \mathbf{u}$$

$$\frac{d}{dt}(\text{momentum}) + \text{net outward momentum flux} = \text{force}$$

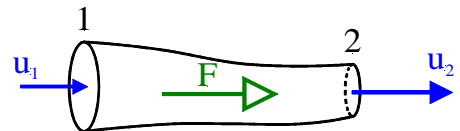
$$\frac{d}{dt}(\rho V \mathbf{u}) + \sum_{\text{faces}} (\rho u_n A) \mathbf{u} = \mathbf{F} \quad (9)$$

(Note that momentum and force are vectors; \mathbf{u} and \mathbf{F} have 3 components).

e.g. for steady quasi-1-d flow:

$$(\text{momentum flux})_{\text{out}} - (\text{momentum flux})_{\text{in}} = \text{force}$$

$$\rho Q(u_2 - u_1) = F$$



Fluid Forces

There are two types:

- *surface forces* (proportional to area; act on control-volume faces)
- *body forces* (proportional to volume)

(i) *Surface forces* are usually expressed in terms of *stress* σ (= force per unit area):

$$\text{force} = \text{stress} \times \text{area}$$

The major surface forces are:

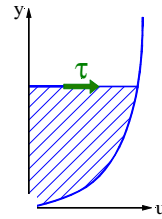
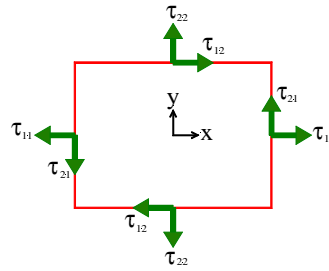
- *pressure* p : always acts normal to a surface

- viscous stresses τ : frictional forces arising from relative motion. For a simple shear flow there is only one non-zero stress component:

$$\tau = \mu \frac{\partial u}{\partial y}$$

but the general expression is more complex:

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right)$$

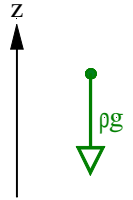


(ii) Body forces

The main body forces are:

- gravity: the force per unit volume is $\rho \mathbf{g} = \rho(0, 0, -g)$

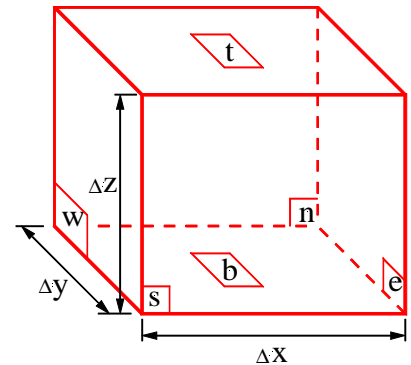
(For constant-density fluids the effects of pressure and weight can be combined in the governing equations as a piezometric pressure $p^* = p + \rho g z$)



- Coriolis forces (in rotating reference frames).

Equivalent Differential Equation

Once again, a conservative differential equation can be derived by considering a fixed cartesian control volume with sides Δx , Δy and Δz .



Using (9) for the x -component, one has:

$$\underbrace{\frac{d}{dt}(\rho V u)}_{\text{rate of change of momentum}} + \underbrace{(\rho u A)_e u_e - (\rho u A)_w u_w + (\rho v A)_n u_n - (\rho v A)_s u_s + (\rho w A)_t u_t - (\rho w A)_b u_b}_{\text{net momentum flux OUT of control volume}} = \underbrace{(p_w A_w - p_e A_e)}_{\text{pressure force in } x \text{ direction}} + \text{viscous and other forces}$$

Substituting the dimensions of the cell:

$$\begin{aligned} \frac{d}{dt}(\rho \Delta x \Delta y \Delta z u) + [(\rho u)_e u_e - (\rho u)_w u_w] \Delta y \Delta z + [(\rho v)_n u_n - (\rho v)_s u_s] \Delta z \Delta x + [(\rho w)_t u_t - (\rho w)_b u_b] \Delta x \Delta y \\ = (p_w - p_e) \Delta y \Delta z + \text{viscous and other forces} \end{aligned}$$

Dividing by the volume, $\Delta x \Delta y \Delta z$ (and changing the order of p_e and p_w for convenience:

$$\frac{\partial(\rho u)}{\partial t} + \frac{(\rho u u)_e - (\rho u u)_w}{\Delta x} + \frac{(\rho v u)_n - (\rho v u)_s}{\Delta y} + \frac{(\rho w u)_t - (\rho w u)_b}{\Delta z} = - \frac{(p_e - p_w)}{\Delta x} + \text{viscous and other forces}$$

Finally, proceeding to the limit as $\Delta x, \Delta y, \Delta z \rightarrow 0$:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} + \frac{\partial(\rho w u)}{\partial z} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \text{other forces} \quad (10)$$

Notes.

(1) The viscous term is given without proof. ∇^2 is the *Laplacian* operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

(2) The pressure force per unit volume in the x direction is given by (minus) the pressure gradient in that direction.

(3) You should be able to derive the y and z -momentum equations similarly by inspection.

(**** MSc course only ****)

Separating surface forces (determined by a stress tensor $\boldsymbol{\sigma}$) and body forces (\mathbf{f} per unit volume), the integral equation for the i component of momentum may be written

$$\frac{d}{dt} \int_V \rho u_i dV + \oint_{\partial V} \rho u_i u_j dA_j = \oint_{\partial V} \sigma_{ij} dA_j + \int_V f_i dV \quad (11)$$

where the stress tensor is given by

$$\sigma_{ij} = -p \delta_{ij} + \tau_{ij}, \quad \tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) \quad (12)$$

For a fixed volume, take d/dt under the integral sign and apply the divergence theorem to the surface integrals:

$$\int_V \left\{ \frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_j} - f_i \right\} dV = 0$$

Since V is arbitrary, the integrand must vanish identically. Hence, for arbitrary surface and body forces:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \quad (13)$$

Splitting the stress tensor into pressure and viscous terms:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i \quad (14)$$

If the fluid is incompressible and viscosity isn't a function of position then the stress term simplifies to give

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + f_i$$

2.2.3 General Scalar

A similar equation may be derived for any physical quantity that is advected or diffused by a fluid flow. For each such quantity an equation is solved for the *concentration* (= amount per unit mass) ϕ : for example, the concentration of salt, sediment or chemical pollutant.

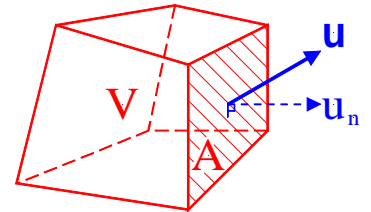
Diffusion occurs when concentration varies with position. It typically occurs from regions of high concentration to regions of low concentration, at a rate proportional to area and the difference in concentration. For many scalars (but see 2.2.5) it may be quantified by *Fick's diffusion law*:

$$\begin{aligned} \text{rate of diffusion} &= -\text{diffusivity} \times \text{gradient} \times \text{area} \\ &= -\Gamma \frac{\partial \phi}{\partial n} A \end{aligned}$$

This is often referred to as *down-gradient diffusion*. Examples include heat diffusion and Darcy's law for flow in porous media.

For an arbitrary control volume:

<i>amount in cell:</i>	$\rho V \phi$	(mass \times concentration)
<i>advective flux:</i>	$(\rho u_n A) \phi$	(mass flux \times concentration)
<i>diffusive flux:</i>	$-\Gamma \frac{\partial \phi}{\partial n} A$	($-\text{diffusivity} \times \text{gradient} \times \text{area}$)
<i>source</i>	$S V$	(source density \times volume)



Balancing the rate of change against the net flux through the boundary and rate of production yields the integral *scalar transport* (or *advection-diffusion*) *equation*:

$$\text{rate of change} + \text{net outward flux} = \text{source}$$

$$\frac{d}{dt} (\rho V \phi) + \sum_{\text{faces}} (\rho u_n A \phi - \Gamma \frac{\partial \phi}{\partial n} A) = S V \quad (15)$$

(**** MSc course only ****)

This may be expressed more mathematically as:

$$\frac{d}{dt} \int_V \rho \phi dV + \oint_{\partial V} (\rho \mathbf{u} \phi - \Gamma \nabla \phi) \cdot d\mathbf{A} = \int_V S dV \quad (16)$$

For a fixed control volume, taking the time derivative under the integral sign and using Gauss's divergence theorem as before gives a corresponding differential form

$$\frac{\partial (\rho \phi)}{\partial t} + \nabla \cdot (\rho \mathbf{u} \phi - \Gamma \nabla \phi) = S \quad (17)$$

2.2.4 Momentum Components as Transported Scalars

The real value of the integral/control-volume approach arises when one realises that, in the momentum equation, if the viscous force $\tau A = \mu \frac{\partial u}{\partial y} A$ is transferred to the LHS it looks like a diffusive flux: e.g. for the x -component of momentum:

$$\frac{d}{dt}(\rho V u) + \sum_{\text{faces}} [(\rho u_n A) u - \mu \frac{\partial u}{\partial n} A] = S'$$

(The viscous force term has been simplified a bit, but the essence of the argument is correct).

Each component of momentum satisfies its own scalar-transport equation with

concentration	\leftrightarrow	velocity (u, v or w)
diffusivity	\leftrightarrow	viscosity μ
source	\leftrightarrow	other forces

Consequently, only one generic scalar-transport equation need be considered.

Computationally, the **same subroutine** may be used to solve the general scalar-transport equation for each variable (but with different diffusivities Γ and source density S).

2.2.5 Non-Gradient Diffusion

The above analysis is just a little simplified. It assumes that fluxes consist only of:


advective flux: $(\rho u_n A) \phi$

simple gradient diffusion: $-\Gamma \frac{\partial \phi}{\partial n} A$

Actually, the real situation is a little more complex. For example, in the u -momentum equation the 1-component of viscous stress through the 2(north)-face is

$$\tau_{12} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

For the u equation the $\partial u / \partial y$ part can be treated as gradient diffusion, but the $\partial v / \partial y$ term can not. The situation is more complex in turbulent flows. In general we must include non-advective fluxes F' that cannot be represented by gradient diffusion. These are discretised conservatively (i.e. worked out on cell faces) but are transferred to the source term as shown:

$$\frac{d}{dt}(\rho V \phi) + \sum_{\text{faces}} [(\rho u_n A) \phi - \mu \frac{\partial \phi}{\partial n} A + F'] = S$$


2.2.6 Moving Control Volumes

The integral conservation equations are equally applicable to **moving** control volumes, provided u_n is interpreted as the normal velocity **relative to the surface**; i.e.

$$u_n = (\mathbf{u}_{\text{flow}} - \mathbf{u}_{\text{surface}}) \cdot \mathbf{n}$$

Crucially, this enables the finite-volume method to be used for calculating flows with moving boundaries: for example, wave motion or flow inside an internal combustion engine.

2.3 Other Differential Equations for Fluid Flow

The finite-volume method is a direct discretisation of the **integral** equations. However, for both theoretical work and notational compactness it is sometimes advantageous to have the governing equations in equivalent **differential** forms. These may be derived directly from the integral relations by applying them to infinitesimal control volumes. Thus:

integral equations describe what is happening to a **finite amount of fluid**;
differential equations describe what is happening at a **point**.

2.3.1 Material Derivatives and Non-Conservative Form of the Flow Equations

There are two equivalent differential forms of the flow equations. Both may be derived from the integral equations by dividing by the volume and taking the limit as this shrinks to zero.

- from **fixed** control volumes we obtain the **conservative** equations (as above);
- using a control volume **moving with the fluid** we obtain **non-conservative** equations.

The conservative differential equations derived in the preceding section are directly equivalent to the control-volume equations. They have derivatives “on the outside” and hence can be integrated directly to give an equivalent integral or control-volume statement of the form

$$flux_{out} - flux_{in} = source$$

For example, in one dimension:

$$\begin{array}{ccc} \frac{d}{dx}(f(x)) = g(x) & \leftrightarrow & f(x_2) - f(x_1) = \int_{x_1}^{x_2} g(x) \, dx \\ \text{differential form} & & \text{integral form} \end{array}$$

(*** MSc only ***)

The three-dimensional version uses partial derivatives and the divergence theorem to change the differentials to surface-integral flux terms.

As an example of how essentially the same equation can appear in conservative and non-conservative forms consider a simple 1-d example:

$$\begin{array}{l} \frac{d}{dx}(y^2) = 2x \quad (\text{conservative form}) \\ 2y \frac{dy}{dx} = 2x \quad (\text{equivalent non-conservative form}) \end{array}$$

The *non-conservative* (or *Lagrangian*) form of the governing differential equations describe how a fixed mass of fluid changes as it moves with the flow.

The non-conservative forms can be derived by either:

- considering a small mass element moving with the flow;
- or
- algebraic manipulation of the conservative form of the flow equations.

Material Derivatives

The rate of change of some property in a fluid element **moving with the flow** is called the *material* (or “total” or “substantive”) derivative. It is denoted by $D\phi/Dt$.

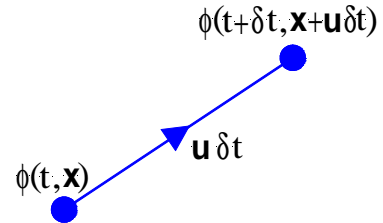
Every field variable ϕ (density, velocity, concentration, ...) is a function of both time and position, $\phi = \phi(t, \mathbf{x})$. If one follows a particular fluid element, then the value of ϕ in that element will change because ϕ is changing:

- with time at a given point ($\partial\phi/\partial t$);
- with position ($\partial\phi/\partial x$ etc.).

Hence the total rate of change as it moves with the flow is

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} \frac{dx}{dt} + \frac{\partial\phi}{\partial y} \frac{dy}{dt} + \frac{\partial\phi}{\partial z} \frac{dz}{dt}$$

i.e.



$\frac{D\phi}{Dt} \equiv \frac{\partial\phi}{\partial t} + u \frac{\partial\phi}{\partial x} + v \frac{\partial\phi}{\partial y} + w \frac{\partial\phi}{\partial z} \quad \text{or, more compactly,} \quad \frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \mathbf{u} \cdot \nabla \phi \quad (18)$

An important example arises when ϕ is a component of velocity. The material derivative is then the *acceleration* $D\mathbf{u}/Dt$. Applying “mass \times acceleration = force” per unit mass of fluid we obtain the momentum equation in *non-conservative* form

$$\underbrace{\rho \frac{Du}{Dt}}_{\text{mass} \times \text{acceleration}} = -\frac{\partial p}{\partial x} + \text{other forces} \quad (19)$$

The LHS of the equation is considerably simpler to write down in this form!

To transform directly from a conservative form of the governing equations to a non-conservative form, expand derivatives (using the product rule) and use continuity to combine the rate-of-change and advection terms. e.g for the general scalar-transport equation:

$$\begin{aligned} & \frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial(\rho v\phi)}{\partial y} + \frac{\partial(\rho w\phi)}{\partial z} \\ &= \frac{\partial\rho}{\partial t}\phi + \rho \frac{\partial\phi}{\partial t} + \frac{\partial(\rho u)}{\partial x}\phi + \rho u \frac{\partial\phi}{\partial x} + \frac{\partial(\rho v)}{\partial y}\phi + \rho v \frac{\partial\phi}{\partial y} + \frac{\partial(\rho w)}{\partial z}\phi + \rho w \frac{\partial\phi}{\partial z} \quad (\text{product rule}) \\ &= \underbrace{\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right]}_{0 \text{ by continuity}} \phi + \rho \underbrace{\left[\frac{\partial\phi}{\partial t} + u \frac{\partial\phi}{\partial x} + v \frac{\partial\phi}{\partial y} + w \frac{\partial\phi}{\partial z} \right]}_{\text{definition of material derivative}} \quad (\text{collecting terms}) \end{aligned}$$

Hence,

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u\phi)}{\partial x} + \frac{\partial(\rho v\phi)}{\partial y} + \frac{\partial(\rho w\phi)}{\partial z} = \rho \frac{D\phi}{Dt}$$

The RHS is obviously easier to write down and is used for notational convenience. However, in the finite-volume method it is the derivatives (\rightarrow differences in an integrated form) on the RHS which are discretised.

2.3.2 Equations for Derived Variables

The basic physical principles of mass, momentum (and energy) conservation can be written down mathematically in many ways. The above forms are general forms in terms of *primitive* variables (i.e. those which can be measured, like velocity or pressure). Sometimes it is more convenient to use *derived* variables. One very common example arises in inviscid flow where it may be shown that all components of velocity can be written in terms of a single scalar variable, the *velocity potential* ϕ :

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

Substituting these into the continuity equation for incompressible flow ($\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$)

gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

or

$$\nabla^2 \phi = 0$$

This is *Laplace's equation* and it is ubiquitous in mathematical physics (e.g. electrostatics, heat transfer, ...). There are many good solvers around that can be used “off-the shelf”. Methods for solving this equation numerically are given in the Computational Mechanics module.

2.4 Compressible and Incompressible Flow

Compressibility is important when **flow-induced variations in pressure or temperature cause significant changes in density**. This can arise in (a) high-speed flow; (b) where there is a significant heat input.

Liquids can almost invariably be treated as incompressible and this is also a good approximation in gases at speeds much less than that of sound ($Ma \ll 1$). Most environmental and civil-engineering flows can be regarded as incompressible.

Density variations can occur for other reasons, notably from salinity variations (oceans) and temperature variations (atmosphere). These are not compressibility effects, but they do lead to *buoyancy forces* (see Section 3 and Dr Lane-Serff's "Geophysical Flows" MSc module).

Compressible Flow

First law of thermodynamics:

$$\text{change of energy} = \text{heat input} + \text{work done on fluid}$$

For compressible flows:

- it is necessary to solve a transport equation for an energy-related variable – usually internal energy e or enthalpy h . From this one can determine absolute temperature T ;
- the mass equation provides a transport equation for ρ ;
- pressure is derived from an *equation of state* such as the ideal-gas law:
$$p = \rho RT \quad (\text{where } e = c_v T)$$
- numerical solution is by *density-based methods*.

Incompressible Flow

By multiplying the momentum equation by velocity and integrating with respect to time we get the *mechanical energy equation*:

$$\text{change of kinetic energy} = \text{work done on fluid}$$

For incompressible flows:

- a separate energy equation is not necessary; (no thermodynamics);
- density is constant along a streamline:

$$\frac{D\rho}{Dt} = 0$$

but may vary between streamlines (e.g. due to salinity changes);

- conservation of mass is replaced by conservation of volume:

$$\sum_{\text{faces}} Q = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- pressure is not derived from a thermodynamic relation but from the need to satisfy simultaneously both mass and momentum equations. We shall see later that this leads to the continuity equation being rephrased as a pressure equation.
- numerical solution is by *pressure-based methods*.

Incompressible flow will be assumed throughout this course.

2.5 Non-Dimensionalisation

Although it is possible to work entirely in dimensional quantities, there are good theoretical reasons for operating with non-dimensional variables. These include the following.

- All *dynamically-similar* problems (i.e. same Reynolds and Froude numbers, etc.) can be solved with a single computation and summarised on a single graph.
- Non-dimensionalisation identifies the relative size of various terms in the governing equations, indicating, for example, which might conveniently be neglected.
- Computational variables are $O(1)$, yielding better numerical accuracy.

For incompressible flow the governing equations are:

$$\text{continuity: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (20)$$

$$x\text{-momentum: } \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u \quad (21)$$

Adopting fundamental reference scales U , L and ρ_0 for velocity, length and density, respectively, and derived scales L/U for time and $\rho_0 U^2$ for pressure, together with a reference pressure p_0 , the dimensional variables can be written in terms of non-dimensional variables (designated by an asterisk *):

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad t^* = \frac{t}{L/U}, \quad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad \rho^* = \frac{\rho}{\rho_0}, \quad p^* = \frac{p - p_0}{\rho_0 U^2}, \quad \text{etc.}$$

Substituting these into mass and x -momentum equations, (20) and (21), yields, respectively

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (22)$$

$$\rho^* \frac{Du^*}{Dt^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}} \nabla^{*2} u^* \quad (23)$$

From this, it is seen that the key dimensionless variable is the *Reynolds number*

$$\text{Re} = \frac{\rho_0 U L}{\mu}$$

If Re is large then viscous forces would be expected to be negligible in much of the flow.

Note.

- Having derived the non-dimensional equations it is usual to drop the asterisks and simply declare “working in non-dimensional variables”.
- In incompressible flow it is only the **differences** of pressure which are important, not its **absolute** value.
- Other dimensionless combinations crop up in other types of flow; for example:
Froude number $\text{Fr} = \frac{U}{\sqrt{gL}}$ in buoyancy-driven flows or flows with a free surface;
Rossby number $\text{Ro} = \frac{U}{\Omega L}$ in rotating flows.

2.6 Summary

- Fluid dynamics is governed by conservation equations for mass, momentum, energy and any additional constituents.
- The governing equations can be written in integral (i.e. control-volume) or differential forms. The latter may be *conservative* (i.e. can be integrated directly to give something of the form “ $flux_{out} - flux_{in} = source$ ”) or *non-conservative*.
- The *finite-volume* method is a direct discretisation of the integral form of the fluid-flow equations.
- The integral equations describe the balance of some quantity within a control volume or “cell”, and take the form:

$$rate\ of\ change + net\ outward\ flux = source$$
- There are really just two canonical equations to discretise and solve:

mass conservation (continuity):

$$\frac{d}{dt}(\rho V) + \sum_{faces} \rho u_n A = 0$$

scalar-transport (or advection-diffusion) equation:

$$\underbrace{\frac{d}{dt}(\rho V \phi)}_{rate\ of\ change} + \underbrace{\sum_{faces} (\rho u_n A \phi)}_{advection} - \underbrace{\Gamma \frac{\partial \phi}{\partial n} A}_{diffusion} = \underbrace{SV}_{sources}$$

$C = \rho u_n A$ is the mass flux through a cell face.

- Each cartesian velocity component (u , v , w) satisfies its own scalar-transport equation. However, these equations differ from those for a passive scalar because they are non-linear and strongly coupled through the advective fluxes and pressure forces.
- For low-speed flows, the energy equation is replaced by an assumption of incompressibility. For incompressible flow the continuity equation becomes conservation of volume:

$$\sum_{faces} Q = 0$$

Taken together with the momentum equation this yields an equation for pressure.

- A non-dimensional form of the governing equations ensures that the main computational variables are $O(1)$, allows a single computation of a class of dynamically-similar flows (those with the same values of Reynolds number, Froude number, Rossby number, etc.) and indicates which are the dominant terms in the governing equations.

