Summary about meshless methods

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Abstract

Some issues usually come up in engineering problems such as large deformation, High-speed impact, dynamic crack growth, etc. When the Finite Element Method (FEM) is used in dealing with these problems, we usually encounter some obstacles because the original grid may have a serious distortions. Meshless methods can avoid the problems above because they use of an approximate program based on discrete points and there is no need to connect them into meshes. The meshless methods have now made a dozen, the application of which include heat conduction, hydrodynamics, structural Mechanics and many other areas and they show highlight superiorities in dealing with issues such as large deformation, High-speed impact, etc. Among them, Smooth Particle Hydrodynamics (SPH), Element-free Galerkin method (EFG) are applied most widely. More over, EFG now is seemed to be an excellent method that has many advantages such as high precision, rapid convergence, no volume lock, etc. This paper makes the Weighted Residual Methods (WRM) by local approximate as the starting point, expounds the theory of meshless methods systematically, summaries and classifications the existing meshless methods and introduces the analysis of ideas and solving process of EFG by a simple example, then, sums up the advantages and shortcomings of meshless methods from mathematical theory and the comparison between meshless and FEM.

Keywords: Meshless methods; Weighted Residual Methods (WRM) by local approximate; Element-free Galerkin method

1 Introduction

The high speed developing technology in the field of computer helps a lot in dealing with the engineering problems which become more and more complex. The conventional method of FEM (Finite Element Method) is widely used because of its high efficiency and feasibility to generate computer program. But sometimes, for example, when come to the issues of large deformation or High-speed impact, using FEM method would be with difficulty. In that situation, the shape of the original grid may change a lot, even a serious distortion. Although the grid can be rebuild, the precision of the computation is influenced much. When simulating dynamic crack growth using the FEM, the gird can not be defined before the computation for the reason that the crack orientation is unknown, so the gird should be rebuild during process. Also the gird generation of some complex models are very difficult. Besides, it is very hard to analyze the problems such as explosion, penetration , incontinuity, precipitous grad using FEM for the gird problems. In order to avoid this, a new method called Meshless which uses an approximate program based on discrete points is introduced and studied.

The method of Meshless, means that there is no need to finish defining gird before giving the system function, at least before approximation amount of field. For a perfect Meshless method, we hope that there is no need to define gird during the whole process. The concept Meshless method focused the attention just after its introduction and soon its advantages when dealing with the problems mentioned ahead emerged. During the past decades, Meshless method have been developing very fast and become more and more an important issue in the field of mechanics. The developing history of Meshless method can be divided into three major phases:

(1) Introduction and development of the Smooth Particle Hydrodynamics (SPH). In 1977, Lucy[1] introduced the method of SPH, which was a kind of pure Lagrange method using no gird. Monaghan[2] studied SPH and he thought the SPH was a kind of kernel approximation method。 The method of SPH was applied in the field of Immense Astrophysics originally. During the phase of the introduction of the SPH, there were some problems such as instability and zero-energy modes, so more and more scholars are trying to improve this method. Dyke[3] and Chen[4] analyzed the instability to find the reason and gave proposals to avoid the that while Vignjevic[5]]studied the zero-energy modes of the system and gave the solution. After ceaselessly improved, the method of SPH has been widely used in the field of hydrodynamic, collision, explosion and high-speed impact when doing the numerical simulating of dynamic reflection。

(2) The application of the method of moving least square (MLS). The original idea of MLS was introduced first by Lancasterand Salkauskas[6] in the 1980s, but until 1992 it was applied in the Diffusion Element Method (DEM) which was introduced by Nayroles[7]]. Later, Belytschko[8]] improved EFG and introduced Element-free Galerkin method (EFG). EFG has obvious advantages in accuracy and convergence efficiency compared to SPH although the high cost of computation, it was widely used.

(3) The flourish of the deployment of Meshless. Along with the introduction of MLS and EFG, the Meshless Method focused more and more attention and its development flourished.During this developing time, new methods introduced as the following: Finite Point Method (FPM) by Onate[9]], Reproducing Kernel Particle Method (RKPM) and multiple-scale RKPM by Liu[10-11], two kinds of Hp-clouds by Oden[12]] 和 Liszka[13]] separately and

Meshless Local Pretrov-Galerkin（MLPG）by Atluri[14], etc. Besides, the optimization of computation, treating the boundary conditions, meshless and finite element, coupled boundary element are hot issues. Many achievements were done by Chinese scholars. In 1995, Professor Zhou Weiheng of Tsinghua University first studied the Meshless method and applied that in the research of fracture mechanics[15]. After that, Lu Wanming and Zhang Xiong[16-17] of Tsinghua University started systematic research of Meshless method, giving the format of Meshless method according to the compact support trial function weighted residual method and the emendation compact support distance function which satisfy maturity conditions。 They also put forward meshless of subdomain type[18], which solve the problem that background grid is needed during the integral process. Long Shuyao[19] of Human University applied the local boundary integral equation method in Elasticity Mechanics and established Meshless Local Pretrov-Galerkin in elasticity mechanics. The Meshless method was used in many domains and showed good effect, which can be seen from the plentiful papers.

There have been about 20 kinds of methods using Meshless so far, which have indubitable advantages when compared to FEM. But as a new generation of computation method in the field of mechanics, Meshless is not mature in a way. In some aspects, for example, rigid mathematical demonstration or large scale application by computer program, Meshless can not be mentioned in the same breath with FEM. Also the Meshless has larger amount of computation than FEM, so Meshless need to be improved ceaselessly.

2 Theoretical foundation of Meshless

2.1 Weighted Residual Method (WRM) [20]

2.1.1 Basic Principles of WRM

Many mechanics problems come down to the solution of differential equations which are controlled by some boundary conditions and initial conditions as follows:

$$F\left[u\left(x,y\right)\right]=0 \quad \text{In domain}\,\Omega$$

$$G\left[u\left(x,y\right)\right]=0 \quad \text{On boundary}\,\Gamma$$

Where F and G are functions in form of differential or partial differential， $u(x,y)$ can be comprehended as a field function in the region Ω. For a complex issue it is difficult to solve the equations above. So we often look for a simpler form of function to close $u(x,y)$ instead of the direct solution of the equations. So we often look for a simpler form of function approaching to $u(x,y)$ instead soluting the equations directly. We choose a approximate function $u^h(x,y)\approx u\left(x,y\right)$, bring it back in the equation. For $u^h(x,y)\neq u\left(x,y\right)$ ， $F\left[u^h\left(x,y\right)\right]\neq 0$ under normal circumstance. We call $R=F\left[u^h\left(x,y\right)\right]$ residual. In order to make the approximate function approach to the original one, we should find a way that eliminates the residual R, that is to attribute to zero.

For the sake of simplicity, take one-dimensional single-variable function for example，

If $F\left[u\left(x\right)\right]=0$ （in the region Ω），

then $\int_\Omega f\left(x\right)F\left[u\left(x\right)\right]dx=0$

Where $f\left(x\right)$ is a random real function.

This formula is called the corresponding weak integral form of $F\left[u\left(x\right)\right]=0$. We can have the residual R eliminated by this way .That is to let

$$\int_\Omega w\left(x\right)F\left[u^h\left(x\right)\right]dx=0.$$

This formula is called Residual Eliminate Function. In the Weighted Residual Method, $u^h\left(x\right)$ is called trial function and $w\left(x\right)$ is called weight function. The accuracy of the solution directly depends on the selection of trial function and weight function. In the WRM based on the entire domain, weight function also represents the discretization scheme. The design schemes of trial function and weight function are introduced followed.

2.1.2 Trial Function

Normally, trial function is expressed as linear combination by a group of basis functions.

$$u^h\left(x\right)=\sum_{i=1}^{n}a_i\cdot p_i\left(x\right),$$

$p_i(x)$ is the basis function, a_i is the coefficient of the basis function。 The mostly used basis functions are polynomials, according to some practical problems, trigonometric functions and exponential functions can be used as basis functions. Otherwise, trial functions also can be structured as differential forms according to taylor expansion, trigonometric series forms according to Fourier expansion and integral forms with in some finite domain. Recently many scholars devoted themselves to the research of introducing the basis functions to improve the precision of algorithms.

There are three kinds of trial functions according to how to structure them: first is the one that satisfies the boundary condition, second is the one satisfies the internal condition, and the last is that satisfies none. Generally, the trial functions for Meshless method are the third kind. It means that solving the unknown value need the control function and also the boundary condition, in another word, to eliminate the residual vector within the domain and on boundary. The fllowing is the common form of the equation:

$$A\int_\Omega w_1\left(x\right)F\left[u^h\left(x\right)\right]dx+B\int_\Gamma w_2\left(x\right)G\left[u^h\left(x\right)\right]dx=0$$

2.1.3 Weight Function

In the weight Residual Method based on the entire domain, different weight functions represent different discretization schemes for differential functions.

(1) Point Collocated Type

Taking thefunction as the weight, we get the Point Collocated precept. We have know the property of δ function as follows

$$\int_{-\infty}^{+\infty} \delta(x-x_j)u(x)dx = u(x_j)$$

In the Residual Eliminate Function, if we let $w(x)=\delta(x-x_j)$,

Then $\int_\Omega \delta(x-x_j)F\left[u^h(x)\right]dx = F\left[u^h(x_j)\right]=0$

The equation above means to make the residual vector zero. For that, we set a series of nodes within the domain, let the residual vector be zero at each node, so we get discrete equations which are not hard to solve.

It is worthy to be noticed that if the trial function is a linear combination of basis functions, the coefficient a_i is unknown and its number equals to the term numbers of the basis functions which compose the trial function. Normally the term numbers of the basis functions can not be too large, so the number of nodes are larger than the number of unknown values, it means that there are redundant unknown values. In this situation, Least Square Principle could be used to get the solution which makes the sum of squares of all equations the least.

⑵ Galerkin Type

Galerkin Type scheme is given by letting the weight functions be the type functions. For example, let the trial function be the followed.

$$u^h(x)=u_1\cdot p_1(x)+u_2\cdot p_2(x)+u_3\cdot p_3(x)+\cdots$$

let $w(x)=p_k(x)$, then the discrete functions is

$$\int_\Omega p_k(x)F\left[\sum_i u_i\cdot p_i(x)\right]dx = 0$$
$$k = 1, 2, 3\cdots$$

These equations have the unique solution because the number of equations equals to the number of unknown values.

⑶ Weighted Least Squares Type

Minimizing $J_R=\int_\Omega R^2dx$, the weighted integral of the square of residual R, then

$$\frac{\partial J_R}{\partial u_j} = 0 \qquad j=1,2,3\cdots\cdots n$$

so $\int_\Omega \frac{\partial R}{\partial u_j}Rdx=0$ and $w(x)=\frac{\partial R}{\partial u_j}$.

2.2 Local Approximate Weighted Residual Method

2.2.1 Local Approximate Process

The difficulty of Weighted Residual Method which has been introduced before is that the obtained coefficient matrix of discrete equations is full matrix. There will be large fluctuation near the boundary and massive computation if the solving domain is large or the number of nodes is large. So in dealing with the practical problems, the Local Approximate Weighted Residual Method is used. Actually FEM as same as Meshless may boil down to the Local Approximate Weighted Residual Method. For example, when using FEM, the solving domain was discretized into massive elements and the displacements of elements are represented by the easy polynomials. Then knot interpolating is carrying on to represent the displacement of each point by the displacement of nodes. This is a kind of local approximate process. Also the process of computing the element stiffness matrix according to the Energy Functional Principle can be deduced by the Galerkin method. The reason for the accurate results by the simple trial function is that solving domain is discretized and the trial function is the approximation of a local element. The Local Approximate Weighted Residual Method is the Meshless Method if the solving domain is discretized by discrete nodes. For the Meshless Method, the crucial thing when carrying on local approximation is the local weight function.

2.2.2 Local Weight Function

Local weight function is different from weight function in WRM said above. It's a kind of function with local influence domain and the trial function is only based on this domain for local approximation. Influence domain is usually rotundity but sometimes ellipse or rectangle. We can comprehend the local weight function a kind of impact among one node to the adjacent ones around. Obviously intensity of the impact will get weaker when the distance between the two nodes increases and can be ignored when the distance is farther than a certain degree. So local weight function of a node should be as follows:

⑴ When x is inside the influence domain, $w(x-x_j)>0$;

outside, $w(x-x_j)=0$。

⑵ In the influence domain $w(x-x_j)$ will get smaller while the distance d between x and the centre of influence domain x_j increases. At the centre point $w(x-x_j)$ obtains the maximum.

⑶ $w(x-x_j)$ reflects the impact among nodes, so it should have some smooth trait.

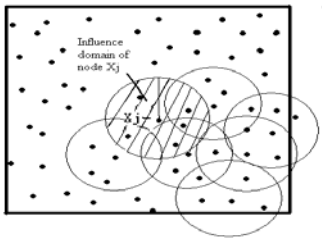

Fig. 2.1 influence domain of a node

When we are researching issues with meshless method, we can do a simulation test and construct the local weight function by interpolating the data of the test. But it is still a problem that needs our research to choose the bulk of the influence domain.

Based on the points above, we often choose subsection bar function or exponential function to be the local weight function, such as

$$w(r)=\begin{cases}1-2r^2 & r\le\frac{1}{2}\\ 2(1-r)^2 & \frac{1}{2}<r\le 1\\ 0 & r>1\end{cases}$$

$$w(d_I)=\begin{cases}\dfrac{e^{-\left(\frac{d_I}{c}\right)^2}-e^{-\left(\frac{d_m}{c}\right)^2}}{1-e^{-\left(\frac{d_m}{c}\right)^2}} & d_I\le d_m\\ 0 & d_I>d_m\end{cases}$$

2.2.3 Trial Function

Under the local approximation, Moving Least Square (MLS) and Kernel approximation are used most often in construction of the trial $u^h(x,x_j)$, which is based on the influence domain of the node x_j.

(1) Moving Least Square Approximation

MLS approximation can be denoted as:

$$u^h(x,x_j)=\sum_{i=1}^{n}p_i(x)\cdot a_i(x),$$

Where $p_i(x)$ is linearity basis function, that is $p(x)=[1,x,x^2,x^3\cdots]$, $a_i(x)$ is the coefficient that relate with coordinate. Usually is ascertained by theories of Best Squares. As follows:

$$L=\sum_j w(x-x_j)\left[\sum_i p_i(x_j)\cdot a_i(x)-u(x_j)\right]^2 \quad (2.2.1)$$

It denotes the best square norm with weight function of the approximation error of each node in the influence domain. Let it take the minimum, in other words let $\frac{\partial L}{\partial a_k}=0$.

Substitute formula 2.1.1 in equation

$$2\sum_j w(x-x_j)\left[\sum_i p_i(x_j)\cdot a_i(x)-u(x_j)\right]p_k(x_j)=0$$

$$k=1,2,3\cdots\cdots n$$

then

$$\sum_i\left[\sum_j w(x-x_j)p_i(x_j)p_k(x_j)\right]a_i(x)=\left[\sum_j w(x-x_j)p_k(x_j)\right]u(x_j)$$

put it into matrix form

$$A(x)a(x)-B(x)u^*=0$$

where $A(x)=P^TW(x)P$, $B(x)=P^TW(x)$

If $A^1(x)$exists, solute $a(x)=A^{-1}(x)B(x)u^*$.

Put it into the formula of trial function,

$$u^h(x,x_j)=\sum_{i=1}^{n}p_i(x)\cdot a_i(x)$$

then

$$u^h(x,x_j)=P^T(x)A^{-1}(x)B(x)u^*=\Phi(x)u^* \quad (2.2.2)$$

Where u^* denotes the numerical value of the nodes, $\Phi(x)$ is called shape function.

(2) Kernel Approximation and Reproducing Kernel Approximation

For δ function,

$$\int_{-\infty}^{+\infty}\delta(x-\bar{x})u(x)dx=u(\bar{x})$$

If we adopt a similar function $\omega(x-x_j)$ instead of δ and truncation it at a proper position, we can construct integral-form trial function as follows

$$u^h(\bar{x},h)=\int_\Omega\omega(x-\bar{x},h)u(x)dx$$

Where $\omega(x-\bar{x})$ is called kernel function, h, called smooth length, is the length of truncation and it is also the radii of influence domain.

If $u(x)$ denotes vector field, we can take grad and divergence operation directly and get the approximate expression of grad, divergence and other physical scalar. Now we make the integral formula discrete, then

$$u^h(\bar{x},h)=\sum_i\omega(x-x_i,h)u(x_i)\Delta X_i=\left[\omega(x)\Delta X\right]^T u^*=\Phi(x)u^* \quad (2.2.3)$$

ΔX_i denotes a length in one-dimensional problem, an area in two and a volume in three. For example in SPH method, ΔX_i denotes volume of the particles where $\Delta X_i=\frac{4}{3}\pi r^3$ or $\Delta X_i=\frac{m_i}{\rho_i}$.

Generally, discrete kernel approximation cannot satisfy the conditions for order consistency and is very unstable near the boundary in dealing with a limited domain issue. Liu et al make some amendment for it as follows

$$u^h(\bar{x},h)=\int_\Omega C(\bar{x},h)\omega(x-\bar{x},h)u(x)dx$$

$C(\bar{x},h)$ is a emendation function that can be reproduced for consistency conditions needed. So this form is called reproducing kernel approximation. It avoids the problem about unsatisfaction of consistency conditions and moreover have the ability of reproducing any form of non-polynomial.

(3) Other approximation scheme

In local approximation, Unit Decomposing, Hp-clouds and Multi Scale Reproducing Kernel Particle Method (MRKPM) are also approximation schemes in common use. Unit Decomposing approximation is constructed by normalizing the MLS; Hp-clouds is also constructed the unit decomposing function by MLS; MRKPM brings in the high resolution power of wavelet analysis and make integral transform by a kind of flexible and adjustable window function which can make detailed analysis to the local part.

2.2.4 Discretization schemes

After getting $u^h(x, x_j)$ by local fitting, there are several ways to solve the problem:

⑴Point Collocated Method which let the equations satisfy the nodes. The main advantages of this method are the high-efficiency computation and no need to integral. But as a result of direct interpolation, Point Collocated Method has problems such as instability in the area among the nodes, also not well accuracy.

⑵ Another solution is that the Element-free Galerkin method which transforms the differential equation to the equivalent form of weak integration. When using EFG, the differential equation was transformed to the form of weak integration which is based on the entire domain, it means that using $u^h(x, x_j)$ by local fitting to approximate the $u^h(x)$ of the entire domain. In order to meet the accuracy, the background grids should be set in the entire domain to do the integral calculation. But the background grid is just background just like its name during the integral process, it has nothing to do with the original-set nodes, and it won't make trouble in dealing with the problem. Meshless Local Pretrov-Galerkin （MLPG）transforms the differential equation to the local form of weak integration, the $u^h(x, x_j)$ by local fitting eliminates the residual only in its influence domain, the integration is with the local area so there is no need to introduce the background grid during the computation. Compared to EFG, the solving process of MLPG is carrying on in divided blocks. At the same time, the coefficient matrix of MLPG is asymmetric so the amount of calculation is relatively large.

⑶ Other methods, such as Subdomain Method, Weighted Least Square Method, Least Square Collocation Method.

2.3 Disposal of boundary condition and coupling of Meshless and FEM [20]

Differing from FEM, trial functions of meshless commonly are not interpolation type so the function's numerical value on the boundary cannot satisfy the real condition. This makes the disposal of boundary condition very intractability for most meshless methods. Now, there are some common methods to deal with the boundary condition. They are Lagrange multiplier, modified variational principles, restrict equation and penalty methods. On the other hand, based on the convenience of FEM, meshless can be coupled to FEM and discrete the regions into the inside meshless domain and boundary FEM domain. But it must be considered of the continuity and harmony conditions on the on the dividing line of the two domains. 。Actually calculation of meshless is more complex than FEM so usually we adopt the coupling way to deal with practical issues, laying out meshless nodes in the central region of impact and finite element in the exterior region of less impact. In this way it can improve the efficiency of computing effectively. Recently there are more and more research done and they help meshless methods more effectively in disposal of boundary conditions.

2.4 Classification of the main meshless method

Based on the introduction above, the theoretical foundation of meshless method is local approximate weighted residual method. We can do meshless analysis by choosing proper schemes for approximation and discretization. Different schemes bring out different meshless methods. All the meshless methods established can be sunned up and classified in this way.

Table 2.1 Classification of the main meshless method

Name of the method	**Discretization scheme**	**Approximation scheme**	**Peculiarity**
Smooth Particle Hydrodynamics（SPH）	*Point Collocated method*	*kernel*	*low efficiency; high-precision*
Hp-Meshless Clouds Method		*Hp-clouds*	
Finite Point Method（FPM）		*MLS*	
Element-free Galerkin method（EFG）	*Galerkin method*	*MLS*	*high efficiency; need of background element for Integral*
Reproducing Kernel Particle Method（RKPM）		*reproducing kernel*	
Partition of Unity Method（PUM）		*partition of unity*	
Hp-clouds		*Hp-clouds*	
Meshless Local Pretrov-Galerkin（MLPG）	*Local Pretrov-Galerkin*	*MLS*	*Free for background element; large calculation*

3 Introduction of several common meshless methods

3.1 *Smooth Particle Hydrodynamics*（SPH）

SPH is a typical particle method. It regards the solution region as one consisted of large number of particles. Nodes are the central point of particle and physical quantities (such as mass) of a particle concentrate at the point. In SPH method, trial function is constructed by kernel approximation which denotes a certain macroscopic physical quantity. The kernel function performs a very important role in SPH method. It can be comprehended as a kind of impact among the adjacent particles that only militates in a certain smooth length. Usually kernel function can be a spline function such as a B- spline like:

$$w(d)=\frac{1}{\pi h^3}\begin{cases}1-6d^2+6d^3 & d\le\frac{1}{2}\\ 2(1-d)^3 & \frac{1}{2}<d\le 1\\ 0 & d>1\end{cases}$$

Where $d=\alpha\|x-x_j\|/r$， α is a parameter，determined by distance of the particles and used for controlling relative weight.

according to formula 2.2.3，

$$u^h(x,x_j,h)=\Phi(x)u^*$$

Replace unknown function $u(x)$ with $u^h(x,x_j,h)$ and substitute it in equation,

$$F\left[u^h(x,x_j,h)\right]=F\left[\Phi(x)u^*\right]=F\left[\Phi(x)\right]u^*=\left[f(x)\right]u^*=0$$

Let the coordinates of the nodes substituted in the equation so that the u^* can be solved out.

In issues of Dynamics, SPH method can simplify the partial differential of spatial variable so that the partial differential equation is transformed to one that only has differential coefficient of time variable.

3. 2 *Element-free Galerkin method*（EFG）

Trial function of EFG adopts MLS approximation and weight function can be an exponential function such as

$$w(d_I)=\begin{cases}\dfrac{e^{-\left(\frac{d_I}{c}\right)^2}-e^{-\left(\frac{d_m}{c}\right)^2}}{1-e^{-\left(\frac{d_m}{c}\right)^2}} & d_I\le d_m\\ 0 & d_I>d_m\end{cases}$$

where d_I is the distance of point x and calculating point x_j， that is $d_I=\|x-x_j\|$， d_m is the radius of x_j 's influence domain. $c=\alpha\cdot\max_{x\in S_j}\|x-x_j\|$ $(1<\alpha<2)$ is a parameter also used for controlling relative weight. S_j denotes the set consisted of the adjacent points around x_j.

Similar as FEM, EFG constructs equilibrium equation by the thin variation of energy functional. Generally speaking form of the equation is as followed

$$\int_\Omega \delta u_i\cdot F\left[u(x)\right]dx=0 \qquad \text{In domain}\,\Omega$$

It can be seen that the weight function here is variation of the trial. Substitute formula 2.2.2 in the equation,then

$$\int_\Omega \Phi_i(x)F\left[\Phi(x)u^*\right]dx=\left(\int_\Omega \Phi_i(x)F\left[\Phi(x)\right]dx\right)u^*=0$$

It's obviously a typical *Galerkin method. Calculating integrals and solve out* u^* then find the approximate solution $u^h(x)$.

For the integral region is the whole solution region, it needs the background element for calculating integral. Usually Gauss method is used to get a high precision.

3.3 Other meshless methods

Reproducing Kernel Particle Method (RKPM) is also a method applied widely now. Compare with SPH, RKPM uses the reproducing kernel approximation and take Galerkin method as the discretization scheme instead of Point Collocated method, which makes the particle method much better in precision and stability. Hp-Meshless Clouds Method and Hp-Clouds Method both use the normalized MLS approximation. The difference between them is that Hp-Meshless Clouds Method takes Point Collocated Method as the discretization scheme but Hp-Clouds Method does Galerkin. MLPG also uses MLS approximation but differing from EFG, MLPG calculates the integral in the subdomains. This makes MLPG a real meshless method that need not background meshes. Enlightened by MLPG, in recent years, many people start to research the transition from background integral to influence domain integral and put forward some now methods such as Finite Field and IMLS.

4 Example

Here quotes an example to introduce the application of EFG method and discuss the precision and convergence rate[21].

Cantilever beam under linearly varying loading.

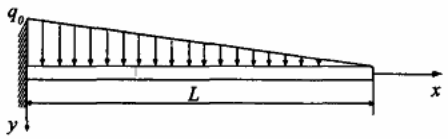

Fig. 4-1 Cantilever beam under linearly varying loading.

Parameters of the ***cantilever beam***：Length L=10，thickness D and width T are both 1，young modulus $E=2\times10^6$ ，Poisson Ratio v=0.3，external load q_0=100。Analytical solution for this issue is

$$w(x)=\frac{q_0}{120EIL}\left(10L^3x^2-10L^2x^3+5Lx^4-x^5\right)$$

Where $I=\frac{DT^3}{12}$. The rotating angle and curvature are

$$\theta(x)=w_x(x)$$

$$\kappa(x)=w_{xx}(x)。$$

Now analyse it with EFG method. The governing equations are:

$$\begin{aligned}&EIw_{xxxx}=q(x),in\Omega=[0,L]\\&w(0)=0,\theta(0)=0\\&M(L)=EIw_{xx}(L)=0\\&V(L)=-EIw_{xxx}(L)=0\end{aligned}\qquad(4.1)$$

equivalent integral weak form of Formula（4.1）is（to look for $w\in S,\forall v\in V$）：

$$\int_0^L v_{xx}EIw_{xx}dx=\int_0^L vqdx-vEIw_{xxx}\Big|_0^L+v_xEIw_{xx}\Big|_0^L \qquad(4.2)$$

S and V are defined as followed:

$$S=\left\{u\,\middle|\,u\in H^2,u=g\quad on\quad x=0\right\}$$

$$V=\left\{v\,\middle|\,v\in H^2,v=0\quad on\quad x=0\right\}$$

It is a homogeneous boundary condition on the left so

$$g=0$$

Now introduce the trial function

$$w^h(x)=\sum_I\Psi_Id_I=\Psi d=\Psi\Lambda^{-T}\bar{d}$$

$$w_x^h(x)=\sum_I\Psi_{I,x}d_I=\Psi_xd=\Psi_x\Lambda^{-T}\bar{d}$$

$$w_{xx}^h(x) = \sum_I \Psi_{I,xx} d_I = \sum_I B_I d_I = B\Lambda^{-T}\overline{d} \quad (4.3)$$

$$v^h(x) = \Psi v = \Psi\Lambda^{-T}\overline{v}$$

$$v_x^h(x) = \Psi_x v = \Psi_x\Lambda^{-T}\overline{v}$$

$$v_{xx}^h(x) = Bv = B\Lambda^{-T}\overline{v} \quad (4.4)$$

Where $\Psi_I = \left\{\Psi_I^d \Psi_I^0\right\}$, $\Psi = A_{I=1}^{NP}(\Psi_I)$, $\Psi_x = A_{I=1}^{NP}\Psi_{I,x}$, $B = A_{I=1}^{NP}(B_I)$ .Substitute formula（4.3）and（4.4）into（4.2）.Now unknown quantity has transformed to the physical quantity of the nodes, so $v^h, v_x^h \in V^h$. According to the randomicity of v^h the discreted equilibrium equation is

$$\overline{K}\overline{d} = \overline{f} \quad (4.5)$$

Where

$$\overline{K} = \Lambda^{-1}K\Lambda^{-T};\ \overline{f} = \Lambda^{-1}f;$$

$$K = A_{I,J=1}^{NP}(K_{IJ});\ f = A_{I=1}^{NP}(f_I)$$

$$K_{IJ} = \int_0^L B_I^T EIB_J d\Omega$$

$$f_I = \int_0^L \Psi_I^T q dx$$

Calculate as follows: cubic basis used; relative influence domain radius is 4; adjacent two nodes make a integral element; 20 Gauss point in each element.

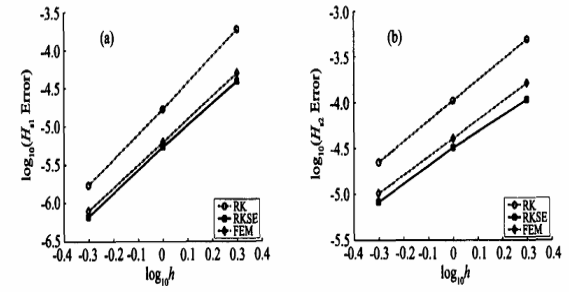

Fig. 4-2 Error comparisons for the cantilever beam problem

（a）rotating angle; （b）curvature

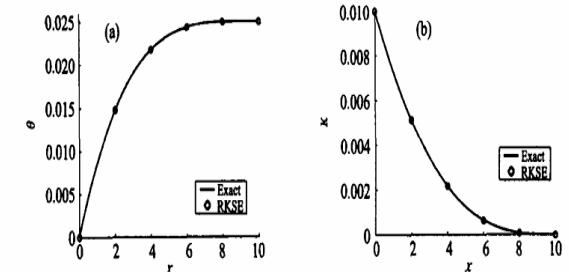

Fig. 4-3 EFG solution with 6 discrete nodes

（a）rotating angle; （b）curvature

Fig 4.2 gives the error and convergence rate, and compares three methods: bivariate MLS approximation（RKSE）, MLS approximation（RK）and Hermitethird-order FEM. It is shown that convergence rate of the three methods are adient but EFG with RKSE has the minimum error. Fig 4.3 is the solution with 6 discrete node**s.** It anastomoses well to the analytical solution.

5 Epilogue

Meshless method is one of the newest results in the Computational Mechanics area and has shown its great advantages in many issues. At the same time it has some shortages of its own, for example the low precision of SPH and the complex calculation process and boundary condition disposal of EFG,etc. Compare with FEM, meshless method has a has a big gap in the theory and applications. However, advantages of meshless method are still the main side so it is worthy for more in-depth study and research. Strict proof of theoretical system and large-scale development of software programs will be the main task for the future of meshless method.

6 References

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