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Analytical solution for wave-induced response of isotropic poro-elastic seabed

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By use of separation of variables, the governing equations describing the Biot consolidation model is firstly transformed into a complex coefficient linear homogeneous ordinary differential equation, and the general solution of the horizontal displacement of seabed is constructed by employing a complex wave number, thus, all the explicit analytical solutions of the Biot consolidation model are determined. By comparing with the experimental results and analytical solution of Yamamoto etc. and the analytical solution of Hsu and Jeng, the validity and superiority of the suggested solution are verified. After investigating the influence of seabed depth on the wave-induced response of isotropic poro-elastic seabed based on the present theory, it can be concluded that the influence depth of wave-induced hydrodynamic pressure in the seabed is equal to the wave length.

linear wave, dynamic water pressure, seabed, analytical solution, Biot consolidation model

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1 Introduction

In recent years, the wave-induced seabed instability problem has attracted much attention from coastal engineers, geotechnical engineers and structural engineers. The main reason is that many coastal structures have been damaged by the wave-induced seabed response rather than from construction deficiencies [1–3]. The wave propagation imposes the dynamical pressure, which makes the pore water pressure and the effective stress within the seabed change, and the excess pore pressure will increase and the vertical effective stress will decrease, which might cause the local seabed instability, even liquefaction. Once the liquefaction phenomenon occurs, the soil grain may be carried off by the bottom ocean current like fluid, or be moved in company with wave action. According to the available investigation information, the seabed instability phenomenon is common, existing in the shallow water, offshore water, even in the deep water. It has been well known that the wave-induced seabed instability is one substantial reason to make the offshore structures damaged or destructed [4].

Based on different assumptions for the compressibility of the seabed and the pore fluid, a number of diverse models for wave-induced seabed instability have been proposed since 1940s, such as uncoupled model (or drained model), consolidation model (or quasi-static model), dynamic model and poro-elastoplastic model [5]. In ocean engineering most of investigations are focused on the former two models. In the uncoupled model, soil is considered as incompressible medium, and the coupling effect of soil motion deformation and pore water penetration is not taken into account, nor the accelerations due to pore water and soil motion. Putnam [6], Sleath [7], Liu [8], Massel [9], Nakamura, Onishi and Minamide [10] and Moshagen and Torum [11] have investigated the wave-induced seabed response by means of this model. According to whether the pore water is compressible or not, the uncoupled model can be separated

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into two types, one is governed by the Laplace's equation (for incompressible pore water), and the other is governed by the diffusion equation (for the compressible pore water). Such solutions for pore pressure are limited to specific cases of soil and wave condition, i.e., the Laplace's equation is applicable to very permeable seabeds such as coarse sand, while the diffusion equation is applicable to poorly permeable seabeds such as clay. On the other hand, in the consolidation model, both pore water and soil are considered to be compressible, but the accelerations due to pore water and soil motion are ignored, and the pore water flow obeys the Darcy's law. This leads to the Biot consolidation equation, which has been widely used to investigate the wave-induced seabed response since 1970s.

Considerable effort has been devoted to analytical solution to the Biot consolidation equation. Yamamoto et al. [12] were the first to develop an approach; Yamamoto [13], and Madsen [14] considered both pore water and soil to be compressible, as well as the wave condition was limited to the progressive wave. A three-dimensional general consolidation equation [15] and a storage equation [16] were introduced in these studies. Among these, Yamamoto et al. [12] considered an isotropic porous seabed of infinite thickness, while Madsen [14] regarded the seabed as a hydraulically anisotropic and unsaturated porous medium of infinite thickness. In addition, Yamamoto [13] investigated the wave-induced response of an isotropic unsaturated seabed of finite thickness, and then, Yamamoto [17] also developed a semi-analytical solution for a nonhomogeneous layered porous seabed in 1981, which was verified by the data collected from the Mississippi Delta. In 1985, Okusa reduced the governing equation in ref. [18] to a fourth-order linear differential equation by use of the compatibility equation under elastic conditions, and developed another analytical solution for the unsaturated seabed of infinite thickness. In the early stage of 1990s, adopting a method similar to Yamamoto et al [12], Hsu and Jeng [19] developed an analytical solution to the dynamic response of a seabed of infinite thickness under a three-dimensional linear short-crested wave. After that they extended this analytical solution to a seabed of finite thickness [20] and obtained explicit expressions, which were reduced to the limited two-dimensional cases of progressive and standing waves. The case for a progressive wave was verified by the semi-analytical solution of Yamamoto et al. [13] for a saturated isotropic seabed and the boundary layer approximate model of Mei and Foda [21] for an unsaturated isotropic seabed. Later, Jeng and Seymour [22, 23] further developed analytical solutions for general soils in the seabed of both infinite and finite thickness. Kitano et al. [24], Kitano and Mase [25] analytically solved the wave-induced seabed response considering the permeability decayed exponentially. Besides the permeability variation, the cross-anisotropy may be reflected in the wave-seabed interaction, which is quite important for the calculation of wave-induced seabed displacement. Recently, considering the wave randomness, Liu and Jeng [26] set up an analytical solution to the random wave-induced soil response within a seabed of infinite thickness, in which the difference on the soil response between regular and random wave loadings was investigated.

In summary, the existing analytical solutions for the Biot consolidation model can be divided into two types, one is for the seabed of finite thickness, and the other is for the seabed of infinite thickness. They can be further divided into several different types according to how the wave and soil characteristics are considered. Regrettably, in order to simplify the derivation process, most of the existing analytical solutions for the Biot consolidation model adopt such an assumption as that the wave-induced excess pore water pressure is equal to zero at the far end of the porous seabed, either for the seabed of infinite thickness or for the seabed of finite thickness. So they are incomplete. In fact, the influence of dynamic hydraulic pressure is limited to a certain depth, although the above assumption can reduce the derivation process, the exact physical significance of the derived solutions may become indistinct. Up to now, the influence depth of wave on the seabed is still not settled, which is quite important for the foundation design of ocean buildings. Therefore, it is necessary to find out one complete analytical solution about wave-seabed interaction. Unlike previous studies, in this study the general solutions for the Biot consolidation equation are searched by use of one general method. Firstly, the governing equations are changed into one sixth-order complex constant coefficient linear homogeneous differential equation concerning the horizontal displacement by use of separation of variables. Secondly, the complex differential equation is solved according to the ordinary derivation method for the linear homogeneous differential equation. Finally, by combining with the given boundary conditions, the general solutions for all the variables are obtained. In addition, a complex wave number is employed so that the obtained analytical solution can take the influence of wave damping into account. Numerical results show that the influence depth of wave-induced hydrodynamic pressure in the seabed is equal to the wave length.

2 Two-dimensional Biot consolidation model

2.1 Wave pressure at the seabed surface

The Cartesian coordinate system for the wave-seabed interaction is shown in Figure 1, where the wave crests are assumed to propagate in the positive *x*-direction, while the *z*-direction is upward from the seabed surface, h is the water depth, and d is the depth of seabed. According to the linear wave theory, the wave surface for any point on the free surface $\eta(x,t)$ can be given by

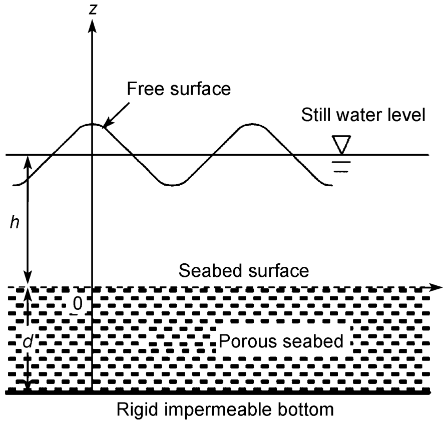

Figure 1 Sketch of wave-seabed interaction.

$$\eta(x,t) = \frac{H}{2}\cos(kx - \mathrm{i}\omega t), \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$; k is a complex wave number, $k = n_r + \mathrm{i}\delta$, $n_r = 2\pi / L$, L is the wavelength; δ is the damping ratio of wave motion; ω is the angular frequency of wave motion; t is the time.

In the linear wave field, the wave pressure is given by

$$p(x,z,t) = -\rho \frac{\partial \phi(x,z,t)}{\partial t}, \tag{2}$$

where ρ is the density of water; $\phi(x,z,t)$ is the velocity potential function of wave field, expressed as below [27]

$$\phi(x,z,t) = -\mathrm{i}\frac{Hg}{2\omega}\frac{\cos k(z+h)}{\sin(kh)}\exp[\mathrm{i}(kx - \omega t)], \tag{3}$$

where g is the gravity acceleration, and h is the water depth.

And then, the wave pressure can be expressed as

$$p(x,z,t) = \frac{\gamma_w H}{2}\frac{\cos k(z+h)}{\cos(kh)}\exp[\mathrm{i}(kx - \omega t)], \tag{4}$$

where γ_w is the unit weight of pore water.

Therefore, the wave pressure at the seabed surface (z=–h) can be given by

$$p = \frac{\gamma_w H/2}{\cos(kh)}\exp[\mathrm{i}(kx - \omega t)]. \tag{5}$$

2.2 Governing equations

Under the wave action, the pore water would flow along the horizontal and vertical directions in the seabed, which could cause the pore water pressure and the effective stresses to change. Assume that the flow of pore water within the seabed obeys the Darcy's law and that the seabed is a hydraulically isotropic poro-elastic medium. Here, the space variability of the permeability of seabed is not considered. According to the two-dimensional consolidation theory of Biot (1941), the consolidation equation can be expressed as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} - \frac{\gamma_w n\beta}{k_z}\frac{\partial p}{\partial t} = \frac{\gamma_w}{k_z}\frac{\partial}{\partial t}(\varepsilon_v), \tag{6}$$

where p is the excess pore water pressure; k_z is the permeability of seabed; n is the soil porosity; ε_v is the volume strain of the porous medium; β is the compressibility of pore water, which is defined as

$$\beta = \frac{1}{K} + \frac{1 - S_r}{p_{w0}}, \tag{7}$$

where K is the true bulk modulus of pore water; S_r is the degree of saturation; p_{w0} is the absolute pore water pressure.

Under the condition of plain strain, the equations of equilibrium can be written as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial z} = \frac{\partial p}{\partial x}, \tag{8}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau}{\partial x} = \frac{\partial p}{\partial z}, \tag{9}$$

where σ_x and σ_z are the effective stresses in the x and z directions, respectively, and τ is the shear stress.

The plane stress-strain relationship is given as below

$$\sigma_x = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \times \left(\frac{\partial u}{\partial x} + \frac{\nu}{(1-\nu)}\frac{\partial w}{\partial z}\right), \tag{10a}$$

$$\sigma_z = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \times \left(\frac{\partial w}{\partial z} + \frac{\nu}{(1-\nu)}\frac{\partial u}{\partial x}\right), \tag{10b}$$

$$\tau = \frac{E}{2(1+\nu)} \times \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{10c}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_z\right), \tag{10d}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = \frac{1-\nu^2}{E}\left(\sigma_z - \frac{\nu}{1-\nu}\sigma_x\right), \tag{10e}$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_z, \tag{10f}$$

where u and w are the soil displacements in the x and z directions, respectively; E is the elastic modulus of soil; ν is the Poisson ratio of soil; ε_x and ε_z are the soil strains in the x and z directions, respectively.

Eqs. (6), (8) and (9) are the three partial differential equations form the governing equations of the two-dimensional Biot consolidation model in terms of three unknown variables, namely, u, w and p, which can be solved in particular boundary conditions. After these three variables are solved, the soil stress conditions also can be determined.

2.3 Boundary conditions

For an isotropic homogeneous seabed, the wave-induced seabed response can be solved according to some appropriate boundary conditions at the seabed surface and at the rigid impermeable bottom. Firstly, at the seabed surface, the vertical effective stress and the shear stress are equal to zero, and the sinusoidal pressure fluctuation exists, i.e.,

$$\text{at } z=0,\quad \begin{cases} \sigma_z = 0, \\ \tau = 0, \\ P_b = R_e\left\{\dfrac{\gamma_w H/2}{\cos(kh)}\exp[\mathrm{i}(kx-\omega t)]\right\}. \end{cases} \tag{11}$$

Secondly, the excess pore water pressure and the soil displacements induced by wave may vary with depth, and finally must vanish. For the seabed of infinite thickness, the boundary conditions at the bottom can be expressed as

$$\text{at } z \to \infty,\quad \begin{cases} u = 0, \\ w = 0, \\ \dfrac{\partial p}{\partial z} = 0. \end{cases} \tag{12}$$

In fact, the excess pore water pressure would vanish at a finite depth of $-d$, so the boundary conditions can be expressed as

$$\text{at } z = -d,\quad \begin{cases} u = 0, \\ w = 0, \\ \dfrac{\partial p}{\partial z} = 0. \end{cases} \tag{13}$$

3 General solutions to seabed response

For the wave-seabed interaction problem, all the physical variables have the same wave number and angular frequency, and the boundary conditions are periodic in both time and space, therefore, it is reasonable to assume that each component can be given as below

$$u = U(z)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{14a}$$

$$w = W(z)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{14b}$$

$$p = P(z)\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{14c}$$

where $U(z)$, $W(z)$ and $P(z)$ are functions of z. Similar to refs. [12] and [20], only the real parts are considered, but unlike the earlier investigations, in this study $U(z)$, $W(z)$ and $P(z)$ are taken as complex functions.

Substituting eq. (14) into the three governing partial differential equations may lead to three simultaneous ordinary differential equations of second order as below

$$\begin{aligned} &\left(-\frac{k_z}{\gamma_w}k^2 + \mathrm{i}\omega n\beta\right)P(z) + \frac{k_z}{\gamma_w}P''(z) = \omega k U(z) - \mathrm{i}W'(z) \\ &-\frac{k^2(2-2v)}{1-2v}U(z) + U''(z) + \frac{\mathrm{i}k}{1-2v}W'(z) = \frac{\mathrm{i}k}{G}P(z) \\ &-k^2 W(z) + \frac{2-2v}{1-2v}W''(z) + \frac{\mathrm{i}k}{1-2v}U'(z) = \frac{P'(z)}{G}. \end{aligned} \tag{15}$$

After some algebraic manipulation, the simultaneous eq. (15) can be transformed into

$$\begin{cases} U^6(z) - k^2(2+k')U^4(z) + k^4(1+2k')U''(z) \\ -k^6 k' U(z) = 0, & \text{(16a)} \\ W'(z) = \alpha_1[U^4(z) - \alpha_2 U''(z) + \alpha_3 U(z)], & \text{(16b)} \\ P(z) = \dfrac{\mathrm{i}kG(2-2v)}{1-2v}U(z) - \dfrac{\mathrm{i}G}{k}U''(z) + \dfrac{G}{1-2v}W'(z), & \text{(16c)} \end{cases}$$

where

$$k' = 1 - \frac{1}{k^2}\frac{\mathrm{i}\omega\gamma_w}{k_z}\left[n\beta + \frac{(1-2v)(1+v)}{E(1-v)}\right], \tag{17}$$

$$\alpha_1 = \frac{(1-v)k_z}{k\omega\gamma_w\left(\dfrac{n\beta}{2} + \dfrac{(1-2v)(1+v)}{E}\right)}, \tag{18}$$

$$\alpha_2 = \frac{2-3v}{1-2v}k^2 - \frac{\mathrm{i}\omega\gamma_w n\beta}{k_z} \times \frac{1-2v}{2(1-v)}, \tag{19}$$

$$\alpha_3 = k^4 k'. \tag{20}$$

Suppose that eq. (16a) possesses the particular solution as follow

$$U(z) = a\mathrm{e}^{\mathrm{i}bz}. \tag{21}$$

Substituting eq. (21) into eq. (16a) renders the final form for the governing equations

$$(b^2+k^2)^2\left\{b^2 + k^2 - \frac{\mathrm{i}\omega\gamma_w}{k_z}\left[n\beta + \frac{(1-2v)(1+v)}{E(1-v)}\right]\right\} = 0. \tag{22}$$

It is easy to obtain six roots of eq. (22) listed as below

$$b = \begin{cases} \pm\mathrm{i}k, \\ \pm\mathrm{i}k, \\ \pm\mathrm{i}\sqrt{\dfrac{\mathrm{i}\omega\gamma_w}{k_z}\left[n\beta + \dfrac{(1-2v)(1+v)}{E(1-v)}\right] - k^2} = \pm\mathrm{i}k\sqrt{k'}. \end{cases} \tag{23}$$

According to the Euler's formula, it can be deduced that $U(z) = a\cos(\mathrm{i}kz)$ and $U(z) = a\sin(\mathrm{i}kz)$ are two basic solutions to eq. (16a), and it is easy to verify that $U(z) = az\cos(\mathrm{i}kz)$, $U(z) = a\cos(\mathrm{i}k\sqrt{k'}z)$, $U(z) = az\sin(\mathrm{i}kz)$ and $U(z) = a\sin(\mathrm{i}k\sqrt{k'}z)$ are another four basic solutions to eq. (16a). Then, the general solution to eq. (16a) can be expressed as

$$u=[c_1U_1(z)+c_2U_2(z)+c_3U_3(z)+c_4U_4(z)+c_5U_5(z)+c_6U_6(z)]\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$
$$=[c_1\cos(\mathrm{i}kz)+c_2z\cos(\mathrm{i}kz)+c_3\cos(\mathrm{i}k\sqrt{k'}z)+c_4\sin(\mathrm{i}kz)+c_5z\sin(\mathrm{i}kz)+c_6\sin(\mathrm{i}k\sqrt{k'}z)]\mathrm{e}^{\mathrm{i}(kx-\omega t)}. \quad (24)$$

Now, it is necessary to verify that $U_1(z),U_2(z),\cdots$, and $U_6(z)$ are linearly independent. According to the mathematical knowledge, if $U_1(z),U_2(z),\cdots$, and $U_6(z)$ are linearly dependent, the Wronskian constituted by them should be equal to zero under any conditions, i.e.,

$$W[U_1(z),U_2(z),\cdots,U_6(z)]=\begin{vmatrix}U_1(z)U_2(z) & \cdots & U_6(z)\\ U_1'(z)U_2'(z) & \cdots & U_6'(z)\\ \vdots\quad\vdots & \cdots & \vdots\\ U_1^5(z)U_2^5(z) & \cdots & U_6^5(z)\end{vmatrix}\equiv 0. \quad (25)$$

By substituting $U_1(z),U_2(z),\cdots$, and $U_6(z)$ into eq. (25), it can be shown that eq. (25) is false. Consequently, $U_1(z)$, $U_2(z),\cdots$, and $U_6(z)$ are linearly independent, and eq. (24) taken as the general solution to eq. (16a) is true.

Substituting eq. (24) into eq. (16b) leads to the vertical displacement of soil as below.

$$\begin{aligned}w=\alpha_1\cdot\Bigg\{&\Bigg[(3c_2k^2+c_4\mathrm{i}k^3+c_5z\mathrm{i}k^3)-\alpha_2\cdot(c_2+c_4\mathrm{i}k+c_5z\mathrm{i}k)\\&-\alpha_3\cdot\left(\frac{c_2}{k^2}+\frac{c_4}{\mathrm{i}k}+\frac{c_5z}{\mathrm{i}k}\right)\Bigg]\cdot\cos(\mathrm{i}kz)\\&+\left(c_6\mathrm{i}k^3k'\sqrt{k'}-\alpha_2c_6\mathrm{i}k\sqrt{k'}-\frac{\alpha_3c_6}{\mathrm{i}k\sqrt{k'}}\right)\cdot\cos(\mathrm{i}k\sqrt{k'}z)\\&-[(c_1\mathrm{i}k^3+c_2z\mathrm{i}k^3-3c_5k^2)-\alpha_2\cdot(c_1\mathrm{i}k+c_2z\mathrm{i}k-c_5)\\&+\alpha_3\cdot\left(\frac{c_1}{\mathrm{i}k}+\frac{c_2z}{\mathrm{i}k}-\frac{c_5}{k^2}\right)\Bigg]\cdot\sin(\mathrm{i}kz)\\&-\left(c_3\mathrm{i}k^3k'\sqrt{k'}-\alpha_2c_3\mathrm{i}k\sqrt{k'}-\frac{\alpha_3c_3}{\mathrm{i}k\sqrt{k'}}\right)\cdot\sin(\mathrm{i}k\sqrt{k'}z)\Bigg\}\mathrm{e}^{\mathrm{i}(kx-\omega t)}.\end{aligned} \quad (26)$$

Substituting eqs. (24) and (26) into eq. (16c) leads to the excess pore pressure as below

$$\begin{aligned}p=-\frac{(1-v)E}{(1+v)(1-2v)}\Bigg\{&\Bigg[\frac{\alpha_1v}{1-v}\cdot(c_1k^4+c_2zk^4+4c_5\mathrm{i}k^3)\\&-\frac{v\alpha_1\alpha_2}{1-v}\cdot(c_1k^2+c_2zk^2+2c_5\mathrm{i}k)+\frac{v\alpha_1\alpha_3}{1-v}\cdot(c_1+c_2z)\\&+c_1\mathrm{i}k+c_2z\mathrm{i}k-\frac{1-2v}{2(1-v)}\cdot c_5+\frac{(1-2v)\alpha_1}{2(1-v)}\cdot\frac{c_5k^3}{\mathrm{i}}\\&-\frac{(1-2v)\alpha_1\alpha_2}{2(1-v)}\cdot\frac{c_5k}{\mathrm{i}}+\frac{(1-2v)\alpha_1\alpha_3}{2(1-v)}\cdot\frac{c_5}{\mathrm{i}k}\\&+\frac{1-2v}{2(1-v)}\cdot(-c_1\mathrm{i}k-c_2\mathrm{i}kz+c_5)+\frac{(1-2v)\alpha_1}{2(1-v)}\\&\cdot(c_1k^4+c_2zk^4+3c_5\mathrm{i}k^3)-\frac{(1-2v)\alpha_1\alpha_2}{2(1-v)}\cdot(c_1k^2+c_2zk^2+c_5\mathrm{i}k)\\&+\frac{(1-2v)\alpha_1\alpha_3}{2(1-v)}\cdot\left(c_1+c_2z-\frac{c_5\mathrm{i}}{k}\right)\Bigg]\cdot\cos(\mathrm{i}kz)\\&+\frac{1}{2(1-v)}(\alpha_1c_3k^4k'^2-\alpha_1\alpha_2c_3k^2k'+\alpha_1\alpha_2c_3-c_3\mathrm{i}k\sqrt{k'})\\&\cdot\cos(\mathrm{i}k\sqrt{k'}z)+\Bigg[\frac{\alpha_1v}{1-v}\cdot(-4c_2\mathrm{i}k^3+c_4k^4+c_5zk^4)-\frac{v\alpha_1\alpha_2}{1-v}\\&\cdot(-2c_2\mathrm{i}k+c_4k^2+c_5zk^2)+\frac{v\alpha_1\alpha_3}{1-v}\cdot(c_4+c_5z)\\&+(c_4\mathrm{i}k+c_5z\mathrm{i}k)+\frac{1-2v}{2(1-v)}\cdot c_2-\frac{(1-2v)\alpha_1}{2(1-v)}\cdot\frac{c_2k^3}{\mathrm{i}}\\&+\frac{(1-2v)\alpha_1\alpha_2}{2(1-v)}\cdot\frac{c_2k}{\mathrm{i}}-\frac{(1-2v)\alpha_1\alpha_3}{2(1-v)}\\&\cdot\frac{c_2}{\mathrm{i}k}-\frac{1-2v}{2(1-v)}\cdot(c_2+c_4\mathrm{i}k+c_5\mathrm{i}kz)\\&-\frac{(1-2v)\alpha_1}{2(1-v)}\cdot(3c_2\mathrm{i}k^3-c_4k^4-c_5zk^4)+\frac{(1-2v)\alpha_1\alpha_2}{2(1-v)}\\&\cdot(c_2\mathrm{i}k-c_4k^2-c_5zk^2)+\frac{(1-2v)\alpha_1\alpha_3}{2(1-v)}\cdot\left(\frac{c_2\mathrm{i}}{k}+c_4+c_5z\right)\Bigg]\\&\cdot\sin(\mathrm{i}kz)+\frac{1}{2(1-v)}(\alpha_1c_6k^4k'^2-\alpha_1\alpha_2c_6k^2k'+\alpha_1\alpha_2c_6-c_6\mathrm{i}k\sqrt{k'})\\&\cdot\sin(\mathrm{i}k\sqrt{k'}z)\Bigg\}\mathrm{e}^{\mathrm{i}(kx-\omega t)}.\end{aligned} \quad (27)$$

Substituting eqs. (24), (26) and (27) into eqs. (13a)–(13c) can derive the effective normal stresses and shear stress of soil as follows

$$\begin{aligned}\sigma_x=\frac{(1-v)E}{(1+v)(1-2v)}\Bigg\{&\Bigg[\frac{\alpha_1v}{1-v}\cdot(c_1k^4+c_2zk^4+4c_5\mathrm{i}k^3)-\frac{v\alpha_1\alpha_2}{1-v}\cdot(c_1k^2+c_2zk^2+2c_5\mathrm{i}k)+\frac{v\alpha_1\alpha_3}{1-v}\cdot(c_1+c_2z)\\&+(c_1\mathrm{i}k+c_2z\mathrm{i}k)\Bigg]\cdot\cos(\mathrm{i}kz)+\left(\frac{\alpha_1v}{1-v}\cdot c_3k^4k'^2-\frac{v\alpha_1\alpha_2}{1-v}\cdot c_1k^2k'+\frac{v\alpha_1\alpha_3}{1-v}\cdot c_3+c_3\mathrm{i}k\sqrt{k'}\right)\cdot\cos(\mathrm{i}k\sqrt{k'}z)\\&+\Bigg[\frac{\alpha_1v}{1-v}\cdot(-4c_2\mathrm{i}k^3+c_4k^4+c_5zk^4)-\frac{v\alpha_1\alpha_2}{1-v}\cdot(-2c_2\mathrm{i}k+c_4k^2+c_5zk^2)+\frac{v\alpha_1\alpha_3}{1-v}\cdot(c_4+c_5z)\\&+(c_4\mathrm{i}k+c_5z\mathrm{i}k)\Bigg]\cdot\sin(\mathrm{i}kz)+\left(\frac{\alpha_1v}{1-v}\cdot c_6k^4k'^2-\frac{v\alpha_1\alpha_2}{1-v}\cdot c_6k^2k'+\frac{v\alpha_1\alpha_3}{1-v}\cdot c_6+c_6\mathrm{i}k\sqrt{k'}\right)\cdot\sin(\mathrm{i}k\sqrt{k'}z)\Bigg\}\mathrm{e}^{\mathrm{i}(kx-\omega t)},\end{aligned} \quad (28)$$

$$\sigma_z = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\{[\alpha_1\cdot(c_1k^4+c_2zk^4+4c_5\mathrm{i}k^3)-\alpha_1\alpha_2\cdot(c_1k^2+c_2zk^2+2c_5\mathrm{i}k)+\alpha_1\alpha_3\cdot(c_1+c_2z)$$
$$+\frac{\nu}{1-\nu}\cdot(c_1\mathrm{i}k+c_2z\mathrm{i}k)]\cdot\cos(\mathrm{i}kz)+(\alpha_1c_3k^4k'^2-\alpha_1\alpha_2c_3k^2+\alpha_1\alpha_3c_3+\frac{\nu}{1-\nu}\cdot c_3\mathrm{i}k\sqrt{k'})\cdot\cos(\mathrm{i}k\sqrt{k'}z)$$
$$+[\alpha_1\cdot(-4c_2\mathrm{i}k^3+c_4k^4+c_5zk^4)-\alpha_1\alpha_2\cdot(-2c_2\mathrm{i}k+c_4k^2+c_5zk^2)+\alpha_1\alpha_3\cdot(c_4+c_5z)+\frac{\nu}{1-\nu}\cdot(c_4ik$$
$$+c_5z\mathrm{i}k)]\cdot\sin(\mathrm{i}kz)+\left(\alpha_1c_6k^4k'^2-\alpha_1\alpha_2c_6k^2+\alpha_1\alpha_3c_6+\frac{\nu}{1-\nu}\cdot c_6\mathrm{i}k\sqrt{k'}\right)\cdot\sin(\mathrm{i}k\sqrt{k'}z)\}\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{29}$$

$$\tau = \frac{E}{2(1+\nu)}\{[(c_2+c_4\mathrm{i}k+c_5\mathrm{i}kz)+\alpha_1\cdot(3c_2\mathrm{i}k^3-c_4k^4-c_5zk^4)-\alpha_1\alpha_2\cdot(c_2\mathrm{i}k-c_4k^2-c_5zk^2)$$
$$-\alpha_1\alpha_3\cdot(\frac{c_2\mathrm{i}}{k}+c_4+c_5z)]\cdot\cos(\mathrm{i}kz)+(c_6\mathrm{i}k\sqrt{k'}-c_6k^4k'^2-\alpha_1\alpha_2c_6k^2k'-\alpha_1\alpha_3c_6)\cdot\cos(\mathrm{i}k\sqrt{k'}z)$$
$$+[(-c_1\mathrm{i}k-c_2\mathrm{i}kz+c_5)+\alpha_1\cdot(c_1k^4+c_2zk^4+3c_5\mathrm{i}k^3)-\alpha_1\alpha_2\cdot(c_1k^2+c_2zk^2+c_5\mathrm{i}k)$$
$$+\alpha_1\alpha_3\cdot\left(c_1+c_2z-\frac{c_5\mathrm{i}}{k}\right)\Bigg]\cdot\sin(\mathrm{i}kz)+(c_3\mathrm{i}k\sqrt{k'}-c_3k^4k'^2-\alpha_1\alpha_2c_3k^2k'-\alpha_1\alpha_3c_3)\cdot\sin(\mathrm{i}k\sqrt{k'}z)\}\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{30}$$

where c_1 to c_6 are six undetermined coefficients which can be determined by the boundary conditions given by eqs. (11) and (13). If c_1 to c_6 are determined, the explicit solution to the wave-induced seabed response would be completely obtained.

As mentioned before, all of the existing analytical solutions to the Biot consolidation model adopt such an assumption as that the wave-induced excess pore water pressure is equal to zero in the infinite depth of seabed. Such an assumption can reduce the wave-seabed interaction problem; however, it results in that the analytical solution can not answer what the exact influence depth of wave on the seabed is. In this study, the above assumption is not used, therefore, the obtained solution is self-contained, and it can precisely reflect the true physical background. Besides, in the earlier investigations, $U(z)$, $W(z)$ and $P(z)$ in eq. (14) are taken as real functions, while the obtained results are complex functions, so the premise and the conclusion contradict each other. Unlike before, in this study they are taken as complex functions from the beginning to the end, therefore, the derivation process is rigorous. Moreover, the derivation clue can be extrapolated to solve the wave-induced response of seabed with orthogonal anisotropy through changing eqs. (12) to (14) into the constitutive equations of orthogonal anisotropic soil.

4 Numerical results and discussions

4.1 Comparison with experimental data and other analytical solutions

As mentioned before, up to now, the acquired analytical solutions to the wave-induced seabed response can be divided into two types, one is for the seabed of infinite thickness and the other is for the seabed of finite thickness, which are represented by the analytical solutions from Yamamoto et al. [12] and Hsu and Jeng [20], respectively. It is worthy to point out that they all used the initial condition, i.e., t=0 and x=0, which would be adopted in this study for easy comparison, unless special remark.

In analysis, two sets of experimental data are used for verification, which are quoted from ref. [12], the corresponding test conditions are: (i) coarse sand, the wave period T is 1.5 s; (ii) fine sand, the wave period is 2.0 s. In this experiment, the basic parameters are: the thickness of soil is 0.5 m, the water depth is 0.9 m, the wave height is 0.2 m; the measured parameters are: the Poisson ratio is 1/3, and the degree of saturation is 98%. Moreover, the rest physical parameters are identified according to the analytical solution of Yamamoto et al. Then, we compare the present solution with the experimental data, and the analytical solutions of Yamamoto et al. [12] and Hsu and Jeng [20].

Figure 2 is the comparison results, and it shows that under the two test conditions, both the present theory and the solution of Yamamoto et al. [12] are close to the test results. Compared with the solution of Hsu and Jeng [20], the present solution is closer to the experimental data. Besides, through solving $\partial p/\partial z$ by use of the suggested solution and the solution of Hsu and Jeng [20], respectively, it can be found that $\partial p/\partial z=0$ at the bottom of seabed for the suggested solution, while $\partial p/\partial z$=–19.6 at the bottom of coarse sand and $\partial p/\partial z$=6890.6 at the bottom of fine sand for the solution of Hsu and Jeng [20]. Therefore, it can be concluded that the analytical solution obtained by Hsu and Jeng is one approximate solution, and it can not exactly satisfy the boundary condition that the seepage at the bottom of seabed is equal to zero. Since some key physical parameters, such as wave length, permeability, elastic modulus, etc., can not exactly be determined, here, we can not judge whether our solutions are closer to the experimental data than Yamamoto et al. [12]. However, their

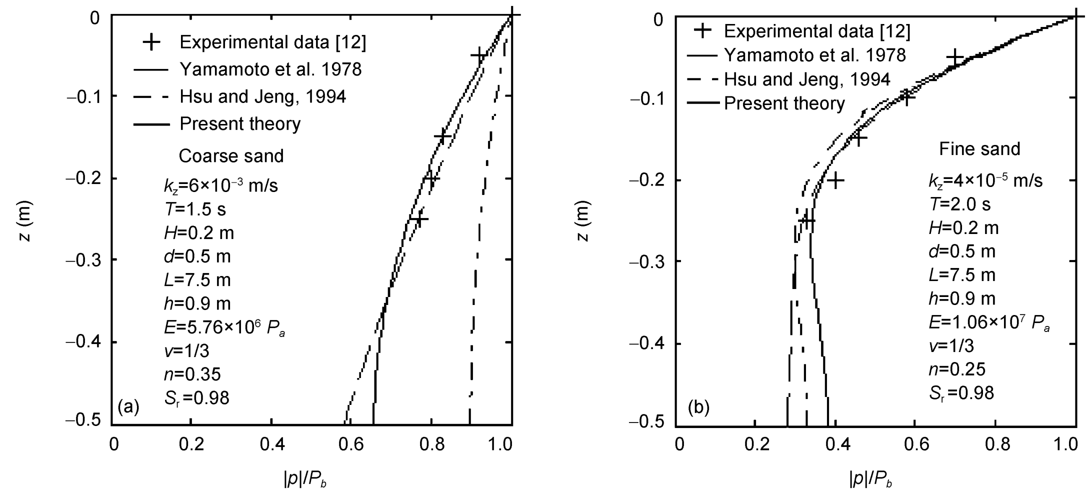

Figure 2 Comparison with experimental data and other analytical solutions.

analytical solution was derived based on the assumption that the seabed thickness is infinite, so it is not exactly applicable to the seabed of finite thickness, while the present theory is applicable to either the seabed of infinite thickness or the seabed of finite thickness.

4.2 Comparison between the present theory and the numerical results

Taking two kinds of homogeneous saturate single-layer soil as the investigated subject, in this section we compare the analytical solution with the numerical results (by use of FlexPDE), in order to verify its validity in a further step. The seabed seepage calculation parameters are given in Table 1, and the comparison results are shown in Figure 3, in which the real lines represent the results of the analytical solution while the dotted lines represent the numerical results. Figure 4 clearly shows that under two different cases, the analytical solutions agree well with the numerical results, so the suggested analytical solution is valid.

Table 1 Seabed seepage calculation parameters

Item	Value	
	Case 1	Case 2
Coefficient of permeability (m/s)	1×10^{-3}	1×10^{-4}
Porosity	0.35	0.35
Saturation level (%)	100	100
Poisson ratio	0.3	0.3
Elastic modulus (Pa)	2×10^{7}	4×10^{7}
Water depth (m)	10	20
Wave height (m)	5	8
Wave period (s)	6	10
Wave length (m)	100	50
Seabed thickness (m)	80	50

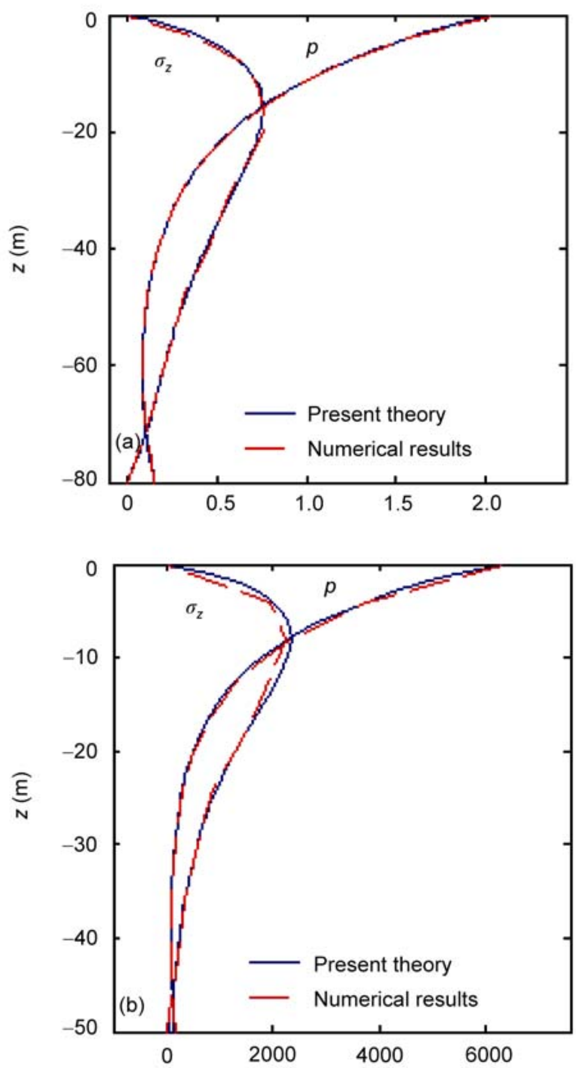

Figure 3 Comparison of the present theory and the numerical results. (a) Case 1; (b) Case 2.

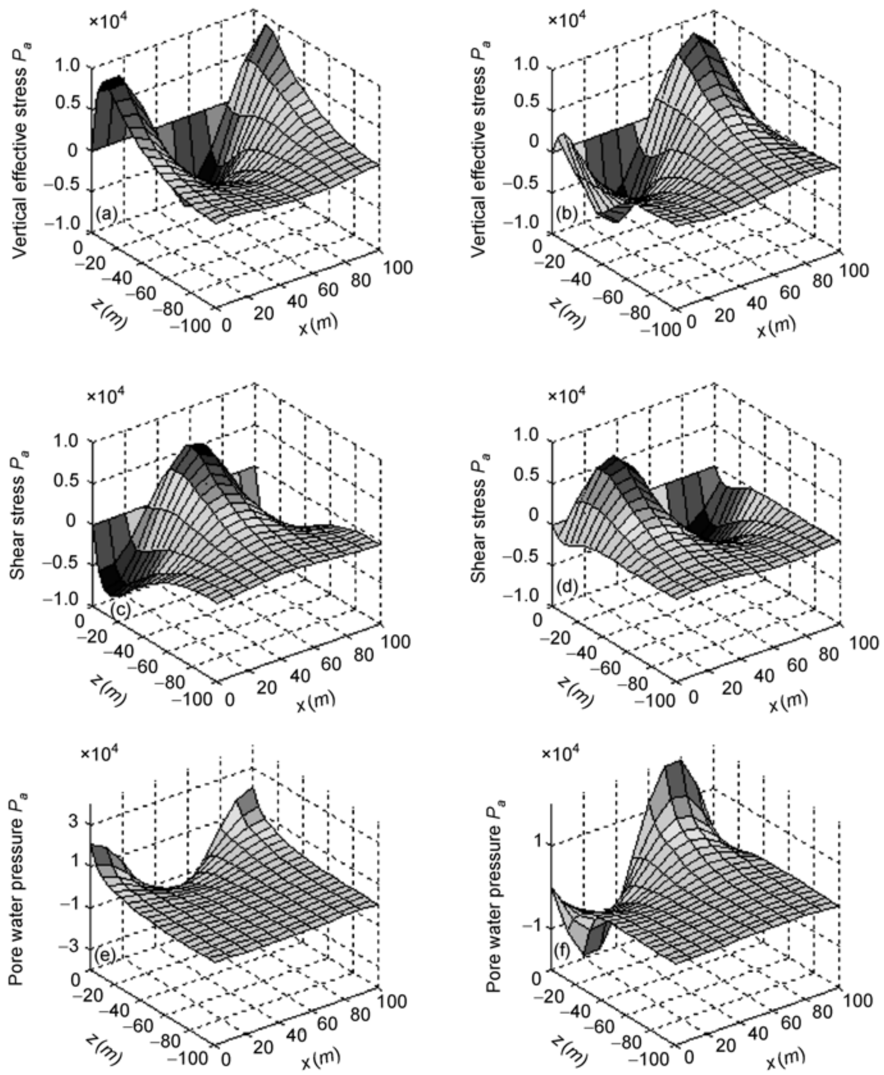

Figure 4 Influence of initial conditions. (a) t=0 s, x=0 m; (b) t=0 s, x=25 m; (c) t=0 s, x=0 m; (d) t=0 s, x=25 m; (e) t=0 s, x=0 m; (f) t=0 s, x=25 m.

5 Several deductions based on the suggested analytical solution

5.1 Influence of initial conditions

As mentioned above, the Biot consolidation model is composed of three linear partial differential equations, whose boundary conditions contain the initial conditions. However, they are not considered in the earlier investigations, and in the above example, such an initial condition as t=0 and x=0 is introduced. In order to investigate the influence of initial conditions, two initial conditions are considered here, namely, (i) t=0, x=0; (ii) t=0, x= 25 m. Under these two initial conditions, the wave-induced seabed responses are calculated by use of the suggested solution. The seabed seepage calculation parameters are given in Table 2, and the calculation results are given in Table 3 and Figure 4. The results show that the initial conditions would affect the calculated seabed responses. Therefore, it is necessary to specify some certain initial conditions. Besides, through comparing the seabed responses under the same initial condition, it is found that the phase of soil vertical effective stress is identical to that of the excess pore water pressure, while the phase difference between the vertical effective stress of soil and the excess pore pressure is equal to π.

Table 2 Seabed seepage calculation parameters

Item	Value
Coefficient of permeability (m/s)	1×10^{-2}
Porosity	0.35
Saturation level (%)	100
Poisson ratio	0.3
Elastic modulus (Pa)	3×10^{7}
Water depth (m)	10
Wave height (m)	5
Wave period (s)	6
Wave length (m)	100
Seabed thickness (m)	100

Table 3 Calculation results

Coefficients	Initial conditions	
	t=0, x=0	t=0, x=25
c_1	$-2.11\times10^{-3}-1.69\times10^{-3}$i	$-1.69\times10^{-3}+2.11\times10^{-3}$i
c_2	$-5.07\times10^{-4}-8.78\times10^{-5}$i	$-8.78\times10^{-4}+5.07\times10^{-5}$i
c_3	$-1.58\times10^{-4}+7.72\times10^{-4}$i	$7.72\times10^{-4}+1.58\times10^{-4}$i
c_4	$-1.70\times10^{-3}+2.12\times10^{-3}$i	$2.12\times10^{-3}+1.70\times10^{-3}$i
c_5	$-8.78\times10^{-5}+5.07\times10^{-4}$i	$5.07\times10^{-5}+8.78\times10^{-4}$i
c_6	$7.73\times10^{-4}+1.58\times10^{-4}$i	$1.58\times10^{-4}-7.73\times10^{-4}$i

5.2 Influence depth of hydrodynamic pressure in the seabed

The influence depth of hydrodynamic pressure in seabed is one of the questions concerned by ocean engineers. However, it has not been settled up to date. In this study, we discuss this question by use of the suggested solution. The seepage parameters are given in Table 2, and seven kinds of seabed thicknesses are considered, the calculated amplitude responses of seabed are shown in Figure 5. Figure 5(a) shows that under the condition d<0.2L (d represents the seabed thickness, and L represents the wave length), the maximal horizontal displacement of soil occurs near the seabed surface, and that under the condition d>0.2L, the wave-induced horizontal displacement of soil firstly increases along with depth, and reaches the maximal value near the depth of 0.2L, then decreases along with depth, finally becomes zero at the bottom (d<L) or at the depth of L(d⩾L). Figure 5(b) shows that the vertical displacement of soil decreases along with depth, the maximal vertical

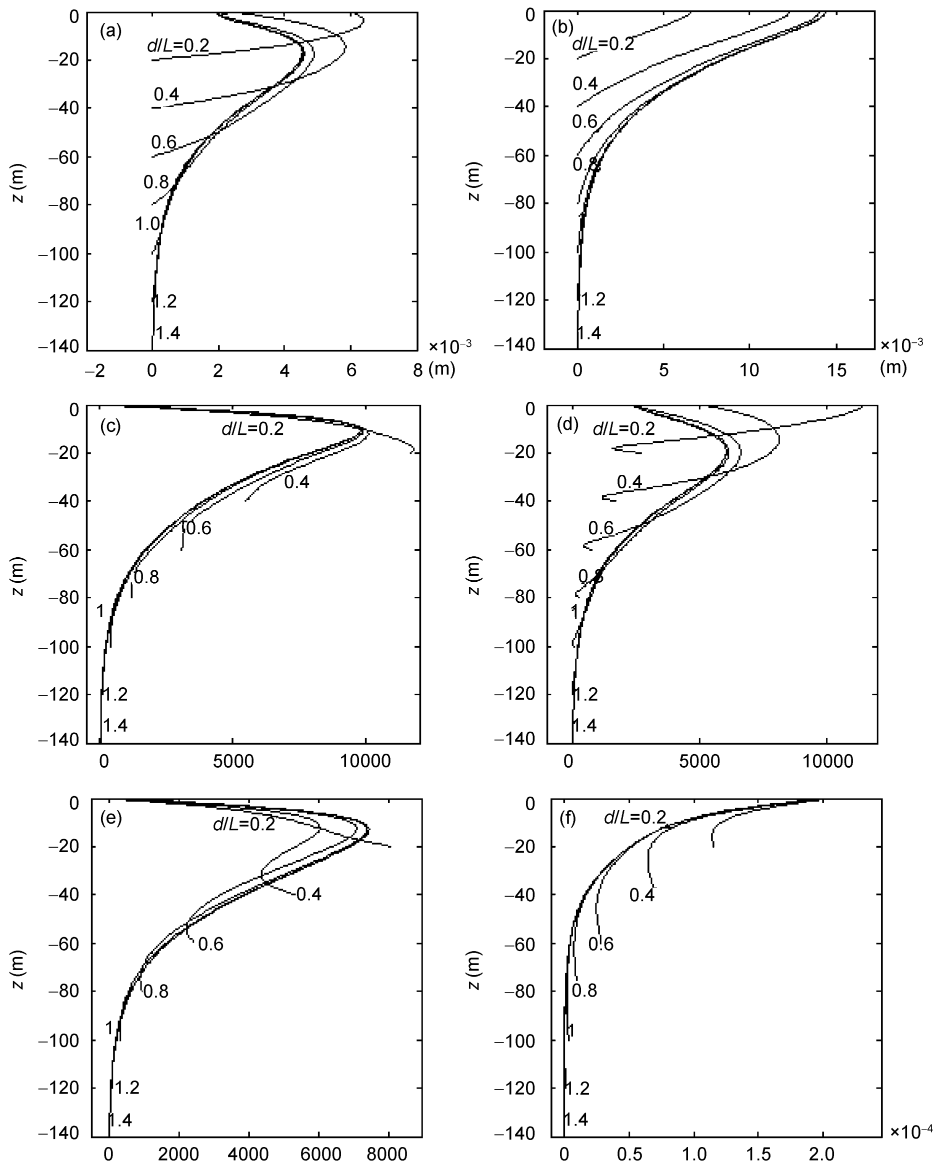

Figure 5 Influence of seabed thickness. (a) Horizontal displacement; (b) vertical displacement; (c) vertical effective stress P_a; (d) horizontal effective stress P_a; (e) shear stress P_a; (f) pore pressure P_a.

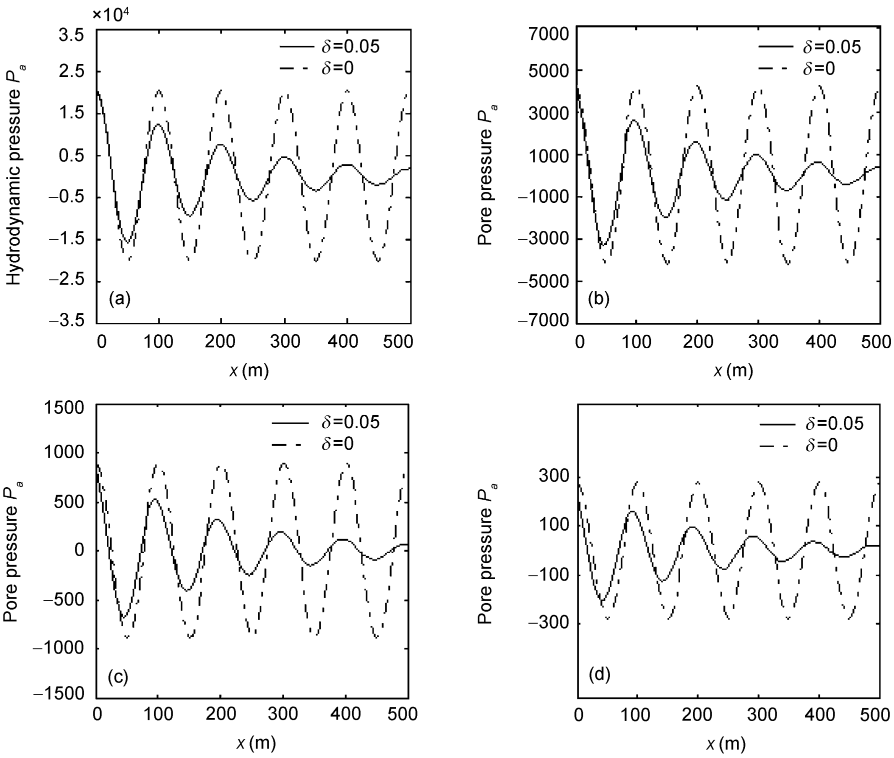

Figure 6 Influence of wave damping. (a) z=0; (b) z=$-L$/4; (c) z=$-L$/2; (d) $-3L$/4.

displacement of soil occurs at the seabed surface under all conditions, and it is affected by the ratio of the thickness of permeable seabed to the wave length. From Figures 5(a) and (b), we can get the following inspirations. In the design of ocean buildings, the thickness of permeable seabed in the field should be taken into account in the selection of design wave length, the wave length corresponding to the maximal wave height can not always guarantee the safety of the designed buildings; besides, considering the influence of wave-induced seabed instability, the embedded depth of ocean pile foundation should not be less than 0.2L. From Figures 5(c)–(e), it can be known that the wave-induced effective stresses of soil firstly increase along with depth, and reach the maximal values near the depth of 0.2L, and then decrease along with depth; besides under the condition d<L, the amplitude values of the horizontal effective stress and the shear stress may slightly increase near the bottom of seabed, which may be caused by the resistance against seepage at the bottom of seabed. Figure 5(f) illustrates that the wave-induced excess pore pressure decreases along with depth, and becomes zero at the depth of one wave length. Therefore, based on the physical background that the excess excess pore water pressure induced by wave must approach to zero at a finite depth, and from the excess pore water pressure results by the suggested solution, it can be concluded that the influence depth of the wave-induced hydrodynamic pressure in the seabed is limited to one wave length. According to such a conclusion, we suggest that in the simulation of wave-induced seabed response by use of numerical methods, if the thickness of permeable seabed is less than the wave length, the calculated thickness should be taken as the actual thickness, while for other cases the calculated thickness should be equal to the wave length.

5.3 Influence of wave damping

The influence of wave damping can be considered in the suggested analytical solution, and such an example is given in Figure 6. The damping ratio of wave height is taken as δ=0.005, the parameters of seabed and wave are as in Table 2.

Figure 6 demonstrates that the amplitudes of the hydrodynamic pressure at the seabed surface and the excess pore water pressure within the seabed are distinctly reduced due to wave damping, and the amplitudes attenuate gradually along the wave propagation direction; the phases of the hydrodynamic pressure at the seabed surface are free from wave damping, while the excess pore pressure would show hysteresis due to wave damping, which gets more and more distinct along with depth. Thus, it can be seen that the influence of wave damping on the seabed response is of prominence, and should be given much attention in the evaluation of wave-induced seabed instability.

6 Conclusions

In this study, based on the Biot consolidation model, a new analytical solution of the isotropic homogeneous seabed was obtained, together with comprehensive comparisons with experimental data and two typical analytical solutions. Then, the influence depth of wave-induced hydrodynamic pressure in the seabed, the influence of wave damping and the limitations of the Biot consolidation model were discussed, respectively. Based on the derivation and discussion presented above, the main conclusions can be summarized as follows.

(i) Compared with the theories of Yamamoto et al. [12], Hsu and Jeng [20], the developed solution does not adopt any assumptions for reduction, so it is precise and self-contained. The comparisons of the present theory, experimental data and another two theories illustrate that the present theory lies between the theory of Yamamoto et al. [12] and that of Hsu and Jeng [20], and both the present theory and the theory of Yamamoto et al. [12] are closer to the experimental data than that of Hsu and Jeng [20]. The theory of Yamamoto et al. [12] is only applicable to the seabed of infinite thickness, while the present theory is not limited.

(ii) Based on the physical background that the wave-induced excess pore water pressure would approach to zero at a finite depth, together with the calculation results by the present theory, the influence depth of the wave-induced hydrodynamic pressure is limited to one wave length. Accordingly, it is suggested that in the simulation of wave-induced seabed response, if the thickness of permeable seabed is less than one wave length, the calculated thickness of seabed should be taken as the actual thickness of seabed, while for other cases, the calculated thickness had better to be taken as one wave length.

(iii) The wave damping has an effect on the amplitude and phase of the excess pore water pressure, so it should be paid much attention to in the evaluation of wave-induced seabed instability.

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