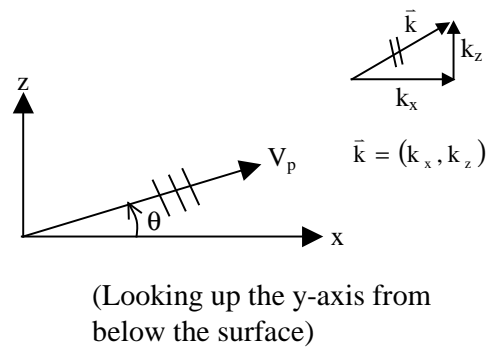


13.021 - Marine Hydrodynamics Lecture 21

6.4 Superposition of Linear Plane Progressive Waves

1. Oblique Plane Waves



Consider wave propagation at an angle θ to the x-axis

$$\eta = A \cos(\overbrace{kx \cos \theta + kz \sin \theta}^{\vec{k} \cdot \vec{x}} - \omega t) = A \cos(k_x x + k_z z - \omega t)$$

$$\phi = \frac{gA}{\omega} \frac{\cosh k(y+h)}{\cosh kh} \sin(kx \cos \theta + kz \sin \theta - \omega t)$$

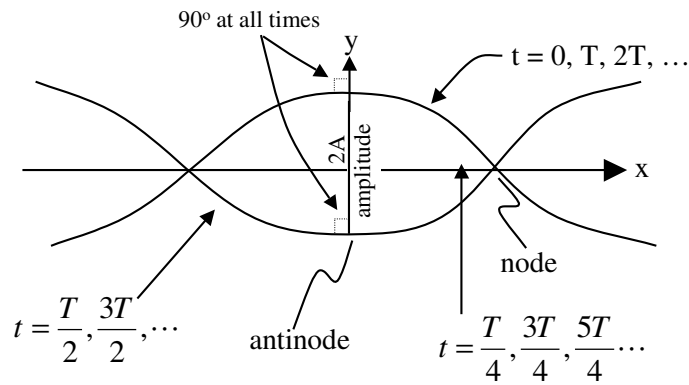
$$\omega^2 = gk \tanh kh; k_x = k \cos \theta, k_z = k \sin \theta, k = \sqrt{k_x^2 + k_z^2}$$

2. Standing Waves



$$\eta = A \cos(kx - \omega t) + A \cos(-kx - \omega t) = 2A \cos kx \cos \omega t$$

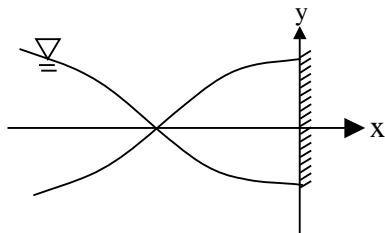
$$\phi = -\frac{2gA \cosh k(y+h)}{\omega \cosh kh} \cos kx \sin \omega t$$



$$\frac{\partial \eta}{\partial x} \sim \frac{\partial \phi}{\partial x} = \dots \sin kx = 0 \text{ at } x = 0, \frac{n\pi}{k} = \frac{n\lambda}{2}$$

Therefore, $\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = 0$. To obtain a standing wave, it is necessary to have perfect reflection at the wall at $x = 0$.

Define the reflection coefficient as $R \equiv \frac{A_R}{A_I} (\leq 1)$.

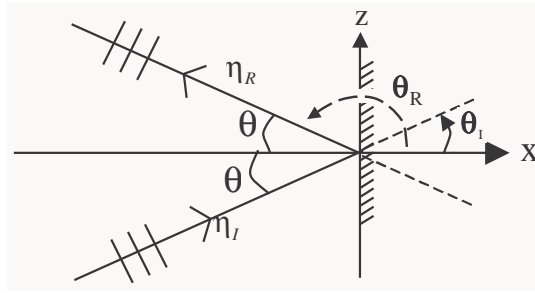


$$A_I = A_R$$

$$R = \frac{A_R}{A_I} = 1$$

3. Oblique Standing Waves

$$\begin{aligned}\eta_I &= A \cos(kx \cos \theta + kz \sin \theta - \omega t) \\ \eta_R &= A \cos(kx \cos(\pi - \theta) + kz \sin(\pi - \theta) - \omega t)\end{aligned}$$



$$\theta_R = \pi - \theta_I$$

Note: same A, R = 1.

$$\eta_T = \eta_I + \eta_R = 2A \underbrace{\cos(kx \cos \theta)}_{\text{standing wave in } x} \underbrace{\cos(kz \sin \theta - \omega t)}_{\text{propagating wave in } z}$$

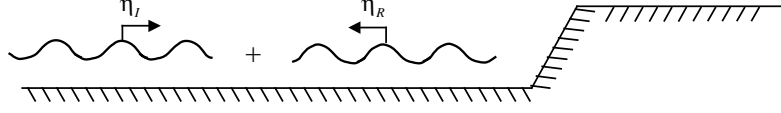
and

$$\lambda_x = \frac{2\pi}{k \cos \theta}; \quad V_{P_x} = 0; \quad \lambda_z = \frac{2\pi}{k \sin \theta}; \quad V_{P_z} = \frac{\omega}{k \sin \theta}$$

Check:

$$\frac{\partial \phi}{\partial x} \sim \frac{\partial \eta}{\partial x} \sim \dots \sin(kx \cos \theta) = 0 \text{ on } x = 0$$

4. Partial Reflection



$$\eta_I = A_I \cos(kx - \omega t) = A_I \operatorname{Re} \{ e^{i(kx - \omega t)} \}$$

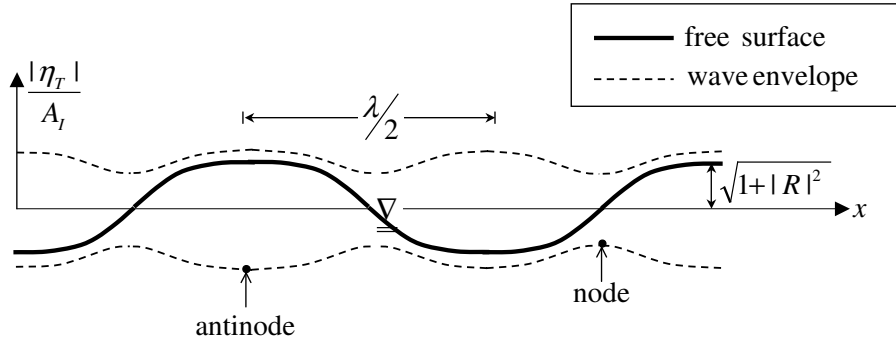
$$\eta_R = A_R \cos(kx + \omega t + \delta) = A_I \operatorname{Re} \{ R e^{-i(kx + \omega t)} \}$$

R : Complex reflection coefficient

$$R = |R| e^{-i\delta}, |R| = \frac{A_R}{A_I}$$

$$\eta_T = \eta_I + \eta_R = A_I \operatorname{Re} \{ e^{i(kx - \omega t)} (1 + R e^{-2ikx}) \}$$

$$|\eta_T|^2 = A_I^2 [1 + |R|^2 + 2|R| \cos(2kx + \delta)]$$



At node,

$$|\eta_T| = |\eta_T|_{\min} = A_I (1 - |R|) \text{ at } \cos(2kx + \delta) = -1 \text{ or } 2kx + \delta = (2n + 1)\pi$$

At antinode,

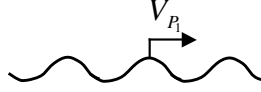
$$|\eta_T| = |\eta_T|_{\max} = A_I (1 + |R|) \text{ at } \cos(2kx + \delta) = 1 \text{ or } 2kx + \delta = 2n\pi$$

$$2kL = 2\pi \text{ so } L = \frac{\lambda}{2}$$

$$|R| = \frac{|\eta_T|_{\max} - |\eta_T|_{\min}}{|\eta_T|_{\max} + |\eta_T|_{\min}} = |R(k)|$$

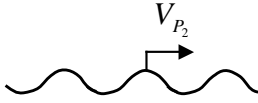
5. Wave Group

2 waves, same amplitude A and direction, but ω and k very close to each other.



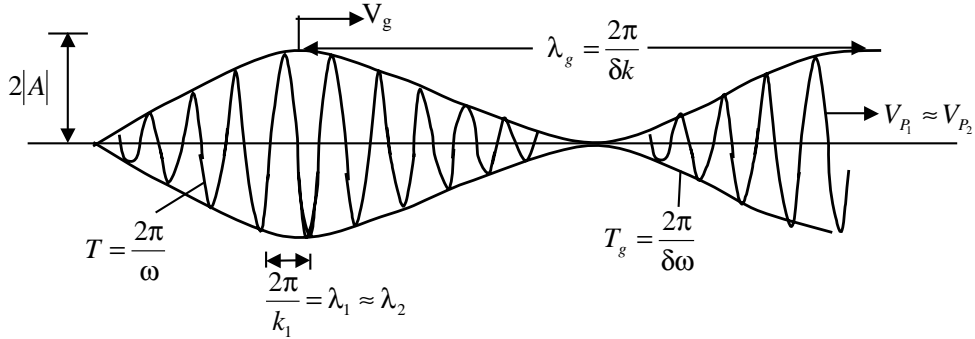
$$\eta_1 = \Re \left(A e^{i(k_1 x - \omega_1 t)} \right)$$

$$\eta_2 = \Re \left(A e^{i(k_2 x - \omega_2 t)} \right)$$



$$\omega_{1,2} = \omega_{1,2}(k_{1,2}) \text{ and } V_{P_1} \approx V_{P_2}$$

$$\eta_T = \eta_1 + \eta_2 = \Re \left\{ A e^{i(k_1 x - \omega_1 t)} \left[1 + e^{i(\delta k x - \delta \omega t)} \right] \right\} \text{ with } \delta k = k_2 - k_1 \text{ and } \delta \omega = \omega_2 - \omega_1$$



$$\left. \begin{aligned} |\eta_T|_{\max} &= 2|A| \text{ when } \delta k x - \delta \omega t = 2n\pi \\ |\eta_T|_{\min} &= 0 \text{ when } \delta k x - \delta \omega t = (2n+1)\pi \end{aligned} \right\} x_g = V_g t, \delta k V_g t - (\delta \omega) t = 0 \text{ then } V_g = \frac{\delta \omega}{\delta k}$$

In the limit,

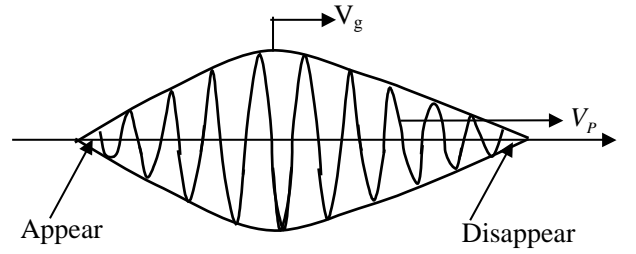
$$\delta k, \delta \omega \rightarrow 0, V_g = \left. \frac{d\omega}{dk} \right|_{k_1 \approx k_2 \approx k},$$

and since

$$\omega^2 = gk \tanh kh \Rightarrow$$

$$V_g = \underbrace{\left(\frac{\omega}{k} \right)}_{V_p} \underbrace{\frac{1}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right)}_n,$$

$$\left. \begin{array}{l} \text{(a) deep water } kh \gg 1 \\ n = \frac{V_g}{V_p} = \frac{1}{2} \\ \text{(b) shallow water } kh \ll 1 \\ n = \frac{V_g}{V_p} = 1 \text{ (no dispersion)} \\ \text{(c) intermediate depth} \\ \frac{1}{2} < n < 1 \end{array} \right\} V_g \leq V_p$$



6.5 Wave Energy - Energy Associated with Wave Motion.

For a single plane progressive wave:

Energy per unit surface area of wave	
• Potential energy PE	• Kinetic energy KE
PE without wave $= \int_{-h}^0 \rho g y dy = -\frac{1}{2} \rho g h^2$	$KE_{wave} = \int_{-h}^{\eta} dy \frac{1}{2} \rho (u^2 + v^2)$
PE with wave $\int_{-h}^{\eta} \rho g y dy = \frac{1}{2} \rho g (\eta^2 - h^2)$	Deep water $= \dots = \underbrace{\frac{1}{4} \rho g A^2}_{KE \text{ const in } x, t}$ to leading order
$PE_{wave} = \frac{1}{2} \rho g \eta^2 = \frac{1}{2} \rho g A^2 \cos^2(kx - \omega t)$	Finite depth $= \dots$
Average energy over one period or one wavelength	
$\overline{PE}_{wave} = \frac{1}{4} \rho g A^2$	$\overline{KE}_{wave} = \frac{1}{4} \rho g A^2$ at any h

- Total wave energy in **deep** water:

$$E = PE + KE = \frac{1}{2} \rho g A^2 \left[\cos^2(kx - \omega t) + \frac{1}{2} \right]$$

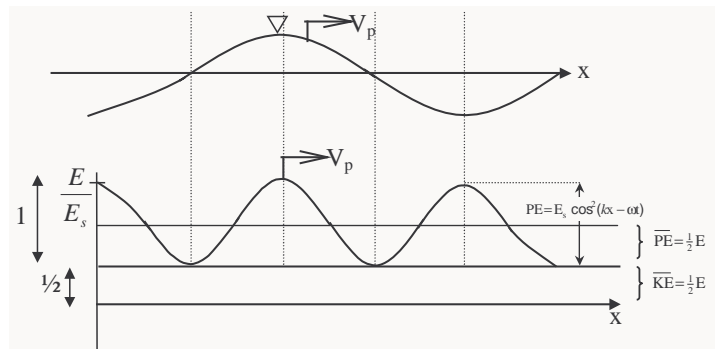
- Average wave energy E (over 1 period or 1 wavelength) for **any** water depth:

$$\overline{E} = \frac{1}{2} \rho g A^2 \left[\underbrace{\frac{1}{2}}_{\overline{PE}} + \underbrace{\frac{1}{2}}_{\overline{KE}} \right] = \frac{1}{2} \rho g A^2 = E_s,$$

$E_s \equiv$ Specific Energy: total average wave energy per unit surface area.

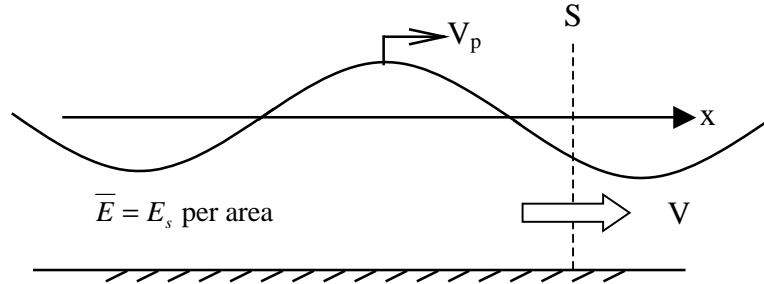
- Linear waves: $\overline{PE} = \overline{KE} = \frac{1}{2} E_s$ (equipartition).

- Nonlinear waves: $\overline{KE} > \overline{PE}$.



$$\text{Recall: } \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

6.6 Energy Propagation - Group Velocity



Consider a fixed control volume V to the right of ‘screen’ S. Conservation of energy:

$$\underbrace{\frac{dW}{dt}}_{\text{rate of work done on S}} = \underbrace{\frac{dE}{dt}}_{\text{rate of change of energy in V}} = \underbrace{\mathfrak{F}}_{\text{energy flux left to right}}$$

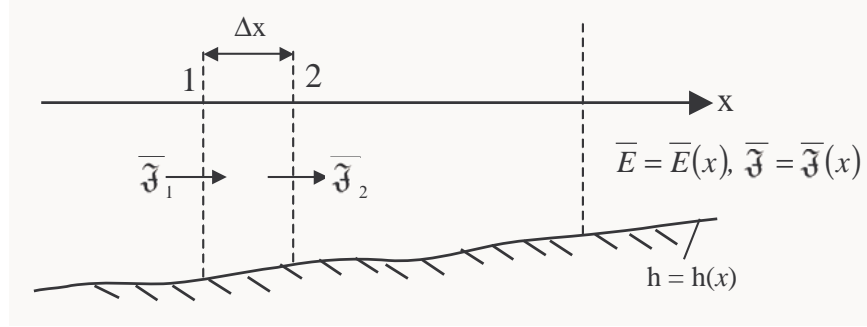
where

$$\mathfrak{F} = \int_{-h}^{\eta} pu \, dy \text{ with } p = -\rho \left(\frac{d\phi}{dt} + gy \right) \text{ and } u = \frac{\partial \phi}{\partial x}$$

$$\bar{\mathfrak{F}} = \underbrace{\left(\frac{1}{2} \rho g A^2 \right)}_{\bar{E}} \underbrace{\left(\frac{\omega}{k} \right)}_{V_p} \underbrace{\left[\frac{1}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right) \right]}_n = \bar{E} (nV_p) = \bar{E} V_g$$

e.g. $A = 3\text{m}$, $T = 10 \text{ sec} \rightarrow \bar{\mathfrak{F}} = 400 \text{KW}/\text{m}$

6.7 Equation of Energy Conservation



$$(\bar{\mathfrak{J}}_1 - \bar{\mathfrak{J}}_2) \Delta t = \Delta \bar{E} \Delta x$$

$$\bar{\mathfrak{J}}_2 = \bar{\mathfrak{J}}_1 + \left. \frac{\partial \bar{\mathfrak{J}}}{\partial x} \right|_1 \Delta x + \dots$$

$$\frac{\partial \bar{E}}{\partial t} + \frac{\partial \bar{\mathfrak{J}}}{\partial x} = 0, \text{ but } \bar{\mathfrak{J}} = V_g \bar{E}$$

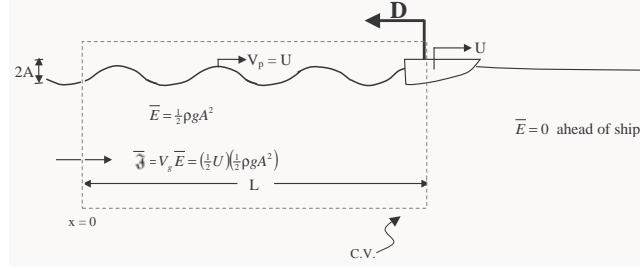
$$\frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial x} (V_g \bar{E}) = 0$$

1. $\frac{\partial \bar{E}}{\partial t} = 0, V_g \bar{E} = \text{constant in } x \text{ for any } h(x).$
2. $V_g = \text{constant (i.e., constant depth, } \delta k \ll k)$

$$\left(\frac{\partial}{\partial t} + V_g \frac{\partial}{\partial x} \right) \bar{E} = 0, \text{ so } \bar{E} = \bar{E}(x - V_g t) \text{ or } A = A(x - V_g t)$$

i.e., wave packet moves at V_g .

6.8 Steady Ship Waves, Wave Resistance



- *Ship wave resistance drag D_w*

Rate of work done = rate of energy increase

$$D_w U + \tilde{\mathfrak{J}} = \frac{d}{dt} (\bar{E} L) = \bar{E} U$$

$$D_w = \frac{1}{U} (\bar{E} U - \overbrace{\bar{E} U / 2}^{\text{deep water}}) = \frac{1}{2} \bar{E} = \frac{1}{4} \rho g A^2 \Rightarrow D_w \propto A^2$$

force / length energy / area

- *Amplitude of generated waves*

The amplitude A depends on U and the ship geometry. Let $\ell \equiv$ effective length.



To *approximate* the wave amplitude A superimpose a bow wave (η_b) and a stern wave (η_s).

$$\eta_b = a \cos(kx) \text{ and } \eta_s = -a \cos(k(x + \ell))$$

$$\eta_T = \eta_b + \eta_s$$

$$A = |\eta_T|_{\max} = 2a \left| \sin\left(\frac{1}{2}k\ell\right) \right| \leftarrow \text{envelope amplitude}$$

$$D_w = \frac{1}{4} \rho g A^2 = \rho g a^2 \sin^2\left(\frac{1}{2}k\ell\right) \Rightarrow D_w = \rho g a^2 \sin^2\left(\frac{1}{2} \frac{g\ell}{U^2}\right)$$

- *Wavelength of generated waves* To obtain the wave length, observe that the *phase speed* of the waves must equal U . For deep water, we therefore have

$$V_p = U \Rightarrow \frac{\omega}{k} = U \xrightarrow{\text{deep water}} \sqrt{\frac{g}{k}} = U, \text{ or } \lambda = 2\pi \frac{U^2}{g}$$

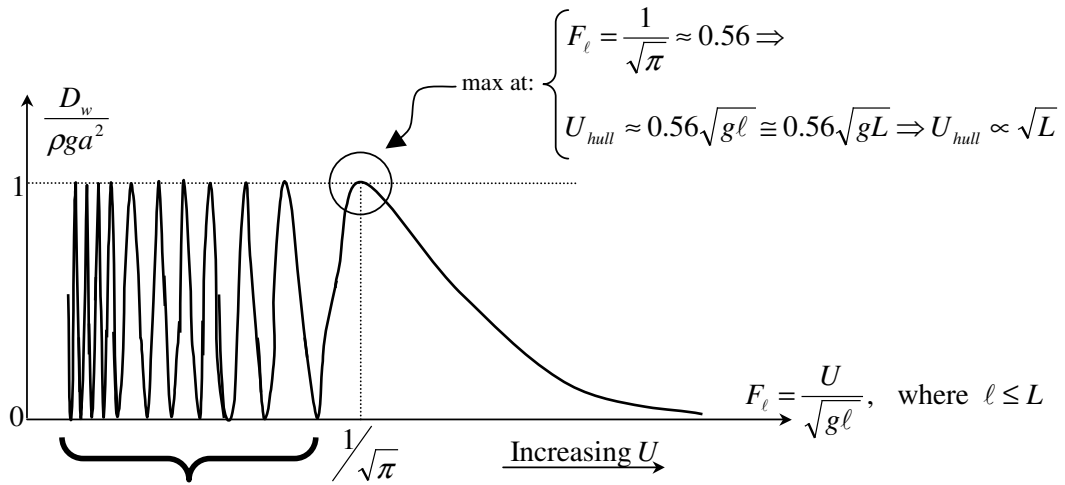
- *Summary* Steady ship waves in deep water.

U = ship speed

$$V_p = \sqrt{\frac{g}{k}} = U; \text{ so } k = \frac{g}{U^2} \text{ and } \lambda = 2\pi \frac{U^2}{g}$$

L = ship length, $\ell \sim L$

$$D_w = \rho g a^2 \sin^2 \left(\frac{1}{2} \frac{g\ell}{U^2} \right) \cong \rho g a^2 \sin^2 \left(\frac{1}{2F_{rL}^2} \right) \cong \rho g a^2 \sin^2 \left(\frac{1}{2F_{rL}^2} \right)$$



Small speed U

- Short waves
- Significant wave cancellation
- $D_w \sim$ small