

13.021 - Marine Hydrodynamics
Lecture 10

3.7 Governing Equations and Boundary Conditions for P-Flow

3.7.1 Governing Equations for P-Flow

(a) Continuity $\boxed{\nabla^2 \phi = 0}$

(b) Bernoulli for P-Flow (steady or unsteady) $\boxed{p = -\rho \left(\phi_t + \frac{1}{2} |\nabla \phi|^2 + gy \right) + C(t)}$

3.7.2 Boundary Conditions for P-Flow

Types of Boundary Conditions:

(c) Kinematic Boundary Conditions - specify the flow velocity \vec{v} at boundaries. $\boxed{\frac{\partial \phi}{\partial n} = U_n}$

(d) Dynamic Boundary Conditions - specify force \vec{F} or pressure p at flow boundary.

$$\boxed{p = -\rho \left(\phi_t + \frac{1}{2} (\nabla \phi)^2 + gy \right) + C(t) \text{ (prescribed)}}$$

The boundary conditions in more detail:

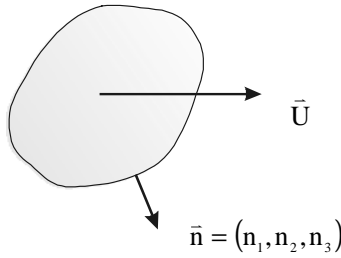
- Kinematic Boundary Condition on an impermeable boundary (no flux condition)

$$\underbrace{\vec{v}}_{\substack{\text{fluid velocity} \\ \vec{v}=\nabla\phi}} \cdot \hat{n} = \underbrace{\vec{U}}_{\text{boundary velocity}} \cdot \hat{n} = \underbrace{U_n}_{\text{normal boundary velocity}} = \text{Given}$$

$$\nabla\phi \cdot \hat{n} = U_n \Rightarrow$$

$$(n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3})\phi = U_n \Rightarrow$$

$$\boxed{\frac{\partial\phi}{\partial n} = U_n}$$



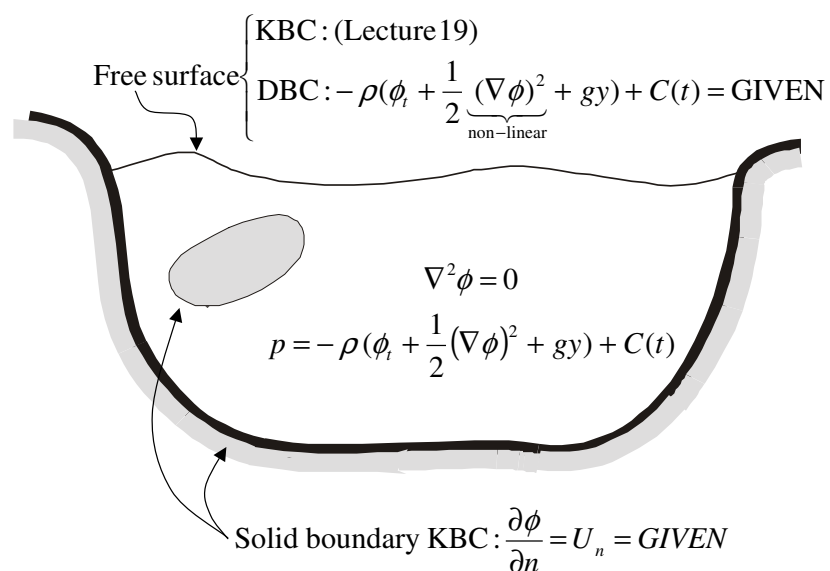
- Dynamic Boundary Condition: In general, pressure is prescribed

$$\boxed{p = -\rho \left(\phi_t + \frac{1}{2} (\nabla\phi)^2 + gy \right) + C(t) = \text{Given}}$$

3.7.3 Summary: Boundary Value Problem for P-Flow

The aforementioned governing equations with the boundary conditions formulate the Boundary Value Problem (BVP) for P-Flow.

The general BVP for P-Flow is sketched in the following figure.

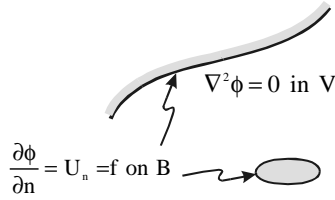


It must be pointed out that this BVP is satisfied **instantaneously**.

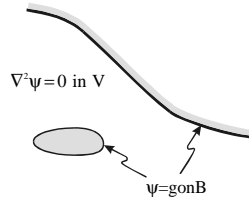
3.8 Linear Superposition for Potential Flow

In the **absence of dynamic boundary conditions**, the potential flow boundary value problem is **linear**.

- Potential function ϕ .



- Stream function ψ .



Linear Superposition: if ϕ_1, ϕ_2, \dots are harmonic functions, i.e., $\nabla^2 \phi_i = 0$, then $\phi = \sum \alpha_i \phi_i$, where α_i are constants, are also harmonic, and is the solution for the boundary value problem provided the kinematic boundary conditions are satisfied, i.e.,

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n} (\alpha_1 \phi_1 + \alpha_2 \phi_2 + \dots) = U_n \quad \text{on } B.$$

The key is to combine known solution of the Laplace equation in such a way as to satisfy the kinematic boundary conditions (KBC).

The same is true for the stream function ψ . The K.B.C specify the value of ψ on the boundaries.

3.8.1 Example

Let $\phi_i(\vec{x})$ denote a unit-source flow with source at \vec{x}_i , i.e.,

$$\begin{aligned}\phi_i(\vec{x}) &\equiv \phi_{\text{source}}(\vec{x}, \vec{x}_i) = \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_i| & (\text{in 2D}) \\ &= - (4\pi |\vec{x} - \vec{x}_i|)^{-1} & (\text{in 3D}),\end{aligned}$$

then find m_i such that

$$\phi = \sum_i m_i \phi_i(\vec{x}) \text{ satisfies KBC on B}$$

Caution: ϕ must be regular for $x \in V$, so it is required that $\vec{x} \notin V$.

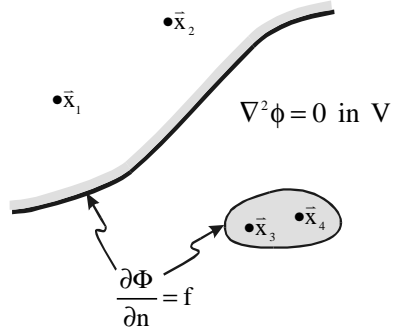


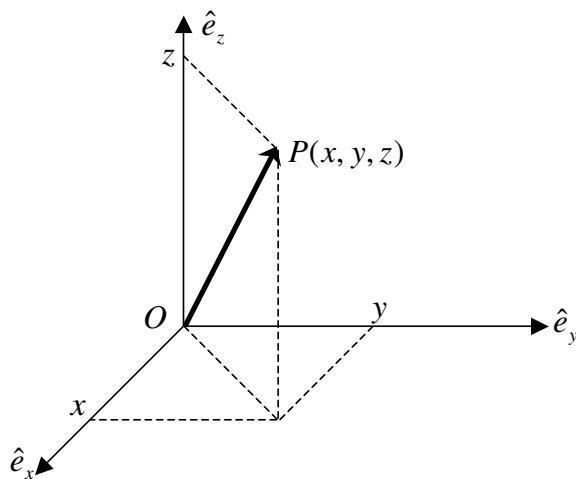
Figure 1: Note: $\vec{x}_j, j = 1, \dots, 4$ are *not* in the fluid domain V .

3.9 - Laplace equation in different coordinate systems (cf Hildebrand §6.18)

3.9.1 Cartesian (x,y,z)

$$\vec{v} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \end{pmatrix} = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$



3.9.2 Cylindrical (r,θ,z)

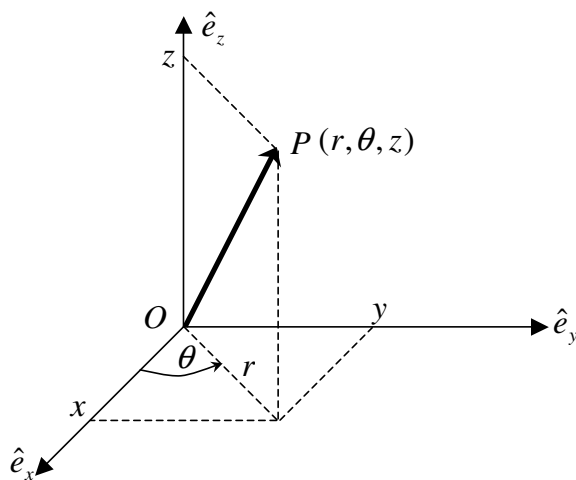
$$r^2 = x^2 + y^2,$$

$$\theta = \tan^{-1}(y/x)$$

$$\vec{v} = \begin{pmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ v_r & v_\theta & v_z \end{pmatrix} = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right)$$

$$\nabla^2 \phi = \underbrace{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}}_{\frac{1}{r} \frac{\partial \phi}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right)} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \Leftrightarrow$$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial \phi}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$



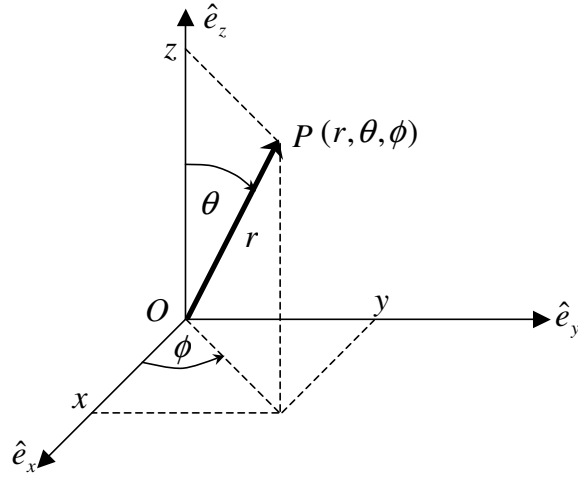
3.9.3 Spherical (r, θ, φ)

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2, \\ \theta &= \cos^{-1}(z/r) \Leftrightarrow z = r (\cos \theta) \\ \varphi &= \tan^{-1}(y/x) \end{aligned}$$

$$\vec{v} = \nabla \phi = \begin{pmatrix} \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r(\sin \theta)} \frac{\partial \phi}{\partial \varphi} \end{pmatrix}$$

$$\nabla^2 \phi = \underbrace{\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r}}_{\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right)} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} \Leftrightarrow$$

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}$$



3.10 Simple Potential flows

1. Uniform Stream $\nabla^2(ax + by + cz + d) = 0$

$$\text{1D:} \quad \phi = Ux + \text{constant} \quad \psi = Uy + \text{constant}; \quad \vec{v} = (U, 0, 0)$$

$$\text{2D:} \quad \phi = Ux + Vy + \text{constant} \quad \psi = Uy - Vx + \text{constant}; \quad \vec{v} = (U, V, 0)$$

$$\text{3D:} \quad \phi = Ux + Vy + Wz + \text{constant} \quad \vec{v} = (U, V, W)$$

2. Source (sink) flow

2D, Polar coordinates

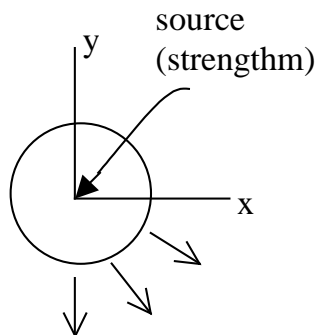
$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \text{ with } r = \sqrt{x^2 + y^2}$$

An axisymmetric solution: $\phi = a \ln r + b$. Verify that it satisfies $\nabla^2 \phi = 0$, except at $r = \sqrt{x^2 + y^2} = 0$. Therefore, $r = 0$ must be excluded from the flow.

Define 2D source of strength m at $r = 0$:

$$\phi = \frac{m}{2\pi} \ln r$$

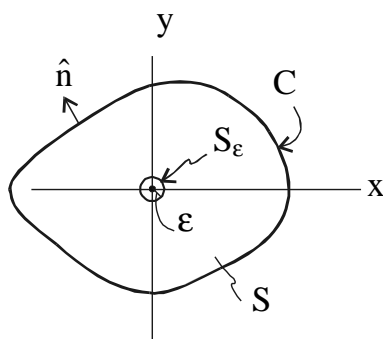
$$\nabla \phi = \frac{\partial \phi}{\partial r} \hat{e}_r = \frac{m}{2\pi r} \hat{e}_r \iff v_r = \frac{m}{2\pi r}, \quad v_\theta = 0$$



Net outward volume flux is

$$\oint_C \vec{v} \cdot \hat{n} ds = \iint_S \nabla \cdot \vec{v} ds = \iint_{S_\varepsilon} \nabla \cdot \vec{v} ds$$

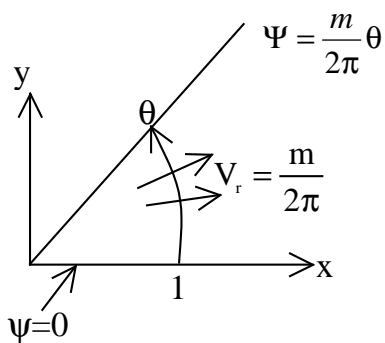
$$\oint_{C_\varepsilon} \vec{v} \cdot \hat{n} ds = \int_0^{2\pi} \underbrace{v_r}_{\frac{m}{2\pi r_\varepsilon}} r_\varepsilon d\theta = \underbrace{m}_{\text{source strength}}$$



If $m < 0 \Rightarrow$ sink. Source m at (x_0, y_0) :

$$\phi = \frac{m}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$\phi = \frac{m}{2\pi} \ln r \text{ (Potential function)} \longleftrightarrow \psi = \frac{m}{2\pi} \theta \text{ (Stream function)}$$



3D: Spherical coordinates

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}, \dots \right), \text{ where } r = \sqrt{x^2 + y^2 + z^2}$$

A spherically symmetric solution: $\phi = \frac{a}{r} + b$. Verify $\nabla^2 \phi = 0$ except at $r = 0$.

Define a 3D source of strength m at $r = 0$. Then

$$\phi = -\frac{m}{4\pi r} \iff v_r = \frac{\partial \phi}{\partial r} = \frac{m}{4\pi r^2}, \quad v_\theta = 0, \quad v_\varphi = 0$$

Net outward volume flux is

$$\oiint v_r dS = 4\pi r_\epsilon^2 \cdot \frac{m}{4\pi r_\epsilon^2} = m \quad (m < 0 \text{ for a sink})$$

3. 2D point vortex

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Another particular solution: $\phi = a\theta + b$. Verify that $\nabla^2\phi = 0$ except at $r = 0$.

Define the potential for a point vortex of circulation Γ at $r = 0$. Then

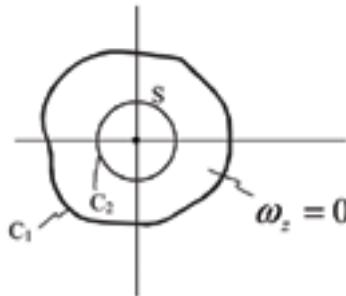
$$\begin{aligned} \phi = \frac{\Gamma}{2\pi}\theta &\iff v_r = \frac{\partial\phi}{\partial r} = 0, \quad v_\theta = \frac{1}{r} \frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r} \text{ and,} \\ \omega_z = \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) &= 0 \text{ except at } r = 0 \end{aligned}$$

Stream function:

$$\psi = -\frac{\Gamma}{2\pi} \ln r$$

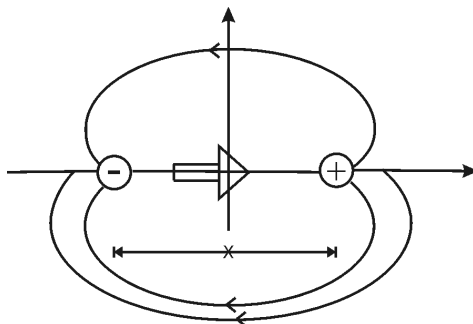
Circulation:

$$\int_{C_1} \vec{v} \cdot d\vec{x} = \int_{C_2} \vec{v} \cdot d\vec{x} + \underbrace{\int_{C_1-C_2} \vec{v} \cdot d\vec{x}}_{\int_S \omega_z dS=0} = \int_0^{2\pi} \frac{\Gamma}{2\pi r} r d\theta = \underbrace{\Gamma}_{\text{vortex strength}}$$



4. Dipole (doublet flow)

A **dipole** is a **superposition** of a **sink** and a **source** with the same strength.

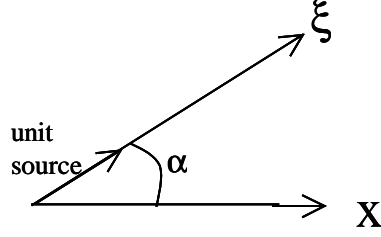


2D dipole:

$$\begin{aligned}\phi &= \frac{m}{2\pi} \left[\ln \sqrt{(x-a)^2 + y^2} - \ln \sqrt{(x+a)^2 + y^2} \right] \\ \lim_{a \rightarrow 0} \phi &= \underbrace{\frac{\mu}{2\pi}}_{\substack{\mu = 2ma \\ \text{constant}}} \frac{\partial}{\partial \xi} \ln \sqrt{(x-\xi)^2 + y^2} \Big|_{\xi=0} \\ &= -\frac{\mu}{2\pi} \frac{x}{x^2 + y^2} = -\frac{\mu}{2\pi} \frac{x}{r^2}\end{aligned}$$

2D dipole (doublet) of moment μ at the origin oriented in the $+x$ direction.

NOTE: dipole $= \mu \frac{\partial}{\partial \xi}$ (unit source)



$$\phi = \frac{-\mu}{2\pi} \frac{x \cos \alpha + y \sin \alpha}{x^2 + y^2} = \frac{-\mu}{2\pi} \frac{\cos \theta \cos \alpha + \sin \theta \sin \alpha}{r}$$

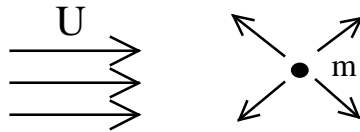
3D dipole:

$$\begin{aligned} \phi &= \lim_{a \rightarrow 0} -\frac{m}{4\pi} \left(\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right) \text{ where } \mu = 2ma \text{ fixed} \\ &= -\frac{\mu}{4\pi} \frac{\partial}{\partial \xi} \frac{1}{\sqrt{(x-\xi)^2 + y^2 + z^2}} \Bigg|_{\xi=0} = -\frac{\mu}{4\pi} \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{\mu}{4\pi} \frac{x}{r^3} \end{aligned}$$

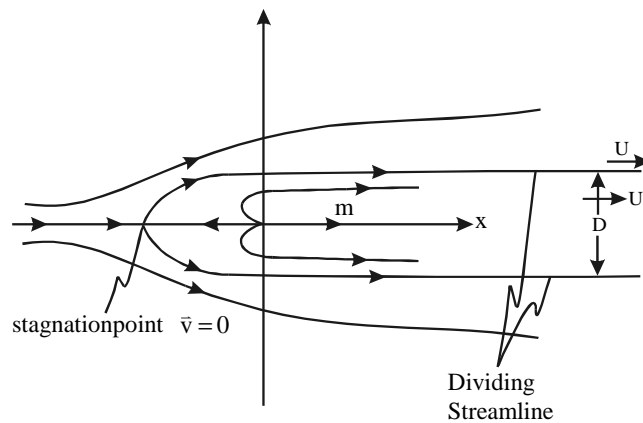
3D dipole (doublet) of moment μ at the origin oriented in the +x direction.

5. Stream and source: Rankine half-body

It is the **superposition** of a **uniform stream** of constant speed U and a **source** of strength m .



2D: $\phi = Ux + \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}$



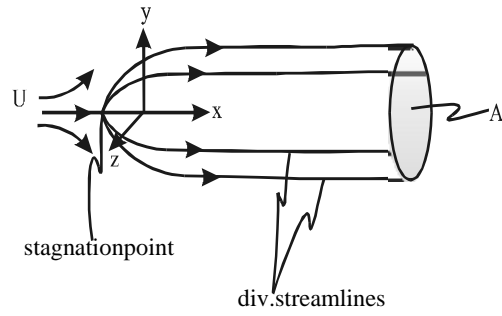
$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{2\pi} \frac{x}{x^2 + y^2}$$

$$u|_{y=0} = U + \frac{m}{2\pi x}, \quad v|_{y=0} = 0 \Rightarrow$$

$$\vec{V} = (u, v) = 0 \text{ at } x = x_s = -\frac{m}{2\pi U}, \quad y = 0$$

For large x , $u \rightarrow U$, and $UD = m$ by continuity $\Rightarrow D = \frac{m}{U}$.

3D: $\phi = Ux - \frac{m}{4\pi\sqrt{x^2 + y^2 + z^2}}$



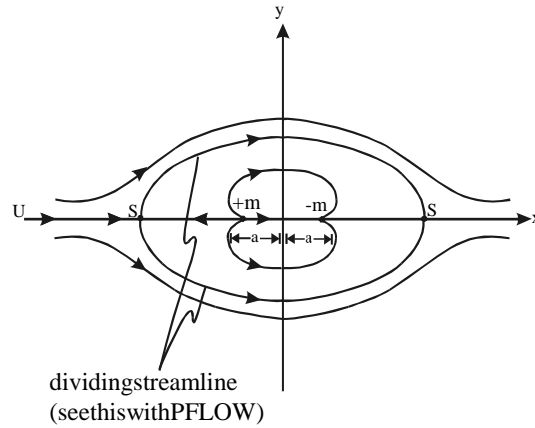
$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi} \frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$u|_{y=z=0} = U + \frac{m}{4\pi} \frac{x}{|x|^3}, \quad v|_{y=z=0} = 0, \quad w|_{y=z=0} = 0 \Rightarrow$$

$$\vec{V} = (u, v, w) = 0 \text{ at } x = x_s = -\sqrt{\frac{m}{4\pi U}}, \quad y = z = 0$$

For large x , $u \rightarrow U$ and $UA = m$ by continuity $\Rightarrow A = \frac{m}{U}$.

6. Stream + source/sink pair: Rankine closed bodies



To have a closed body, a necessary condition is to have $\sum m_{\text{in body}} = 0$

2D Rankine ovoid:

$$\phi = Ux + \frac{m}{2\pi} \left(\ln \sqrt{(x+a)^2 + y^2} - \ln \sqrt{(x-a)^2 + y^2} \right) = Ux + \frac{m}{4\pi} \ln \left(\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \right)$$

3D Rankine ovoid:

$$\phi = Ux - \frac{m}{4\pi} \left[\frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} \right]$$

For Rankine Ovoid,

$$\begin{aligned}
 u = \frac{\partial \phi}{\partial x} &= U + \frac{m}{4\pi} \left[\frac{x+a}{((x+a)^2 + y^2 + z^2)^{3/2}} - \frac{x-a}{((x-a)^2 + y^2 + z^2)^{3/2}} \right] \\
 u|_{y=z=0} &= U + \frac{m}{4\pi} \left[\frac{1}{(x+a)^2} - \frac{1}{(x-a)^2} \right] \\
 &= U + \frac{m}{4\pi} \frac{(-4ax)}{(x^2 - a^2)^2} \\
 u|_{y=z=0} &= 0 \text{ at } (x^2 - a^2)^2 = \left(\frac{m}{4\pi U} \right) 4ax
 \end{aligned}$$

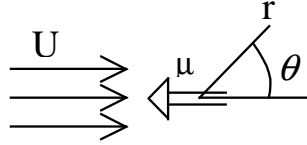
At $x = 0$,

$$u = U + \frac{m}{4\pi} \frac{2a}{(a^2 + R^2)^{3/2}} \text{ where } R = y^2 + z^2$$

Determine radius of body R_0 :

$$2\pi \int_0^{R_0} u R dR = m$$

7. Stream + Dipole: circles and spheres



2D: $\phi = Ux + \frac{\mu x}{2\pi r^2} \underset{x=r \cos \theta}{=} \cos \theta \left(Ur + \frac{\mu}{2\pi r} \right)$

The radial velocity is then

$$u_r = \frac{\partial \phi}{\partial r} = \cos \theta \left(U - \frac{\mu}{2\pi r^2} \right).$$

Setting the radial velocity $v_r = 0$ on $r = a$ we obtain $a = \sqrt{\frac{\mu}{2\pi U}}$. This is the K.B.C. for a stationary circle of radius a . Therefore, for

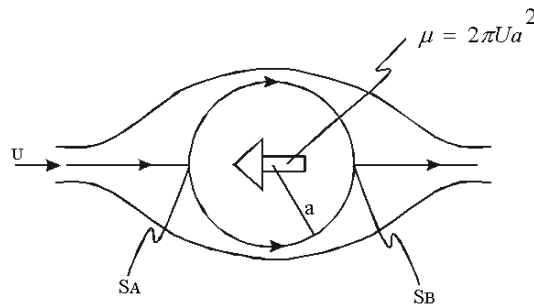
$$\mu = 2\pi U a^2$$

the potential

$$\phi = \cos \theta \left(Ur + \frac{\mu}{2\pi r} \right)$$

is **the** solution to ideal flow past a circle of radius a .

- *Flow past a circle* (U, a) .



$$\begin{aligned}
\phi &= U \cos \theta \left(r + \frac{a^2}{r} \right) \\
V_\theta &= \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \sin \theta \left(1 + \frac{a^2}{r^2} \right) \\
V_\theta|_{r=a} &= -2U \sin \theta \begin{cases} = 0 \text{ at } \theta = 0, \pi & \text{-- stagnation points} \\ = \mp 2U \text{ at } \theta = \frac{\pi}{2}, \frac{3\pi}{2} & \text{-- maximum tangential velocity} \end{cases}
\end{aligned}$$

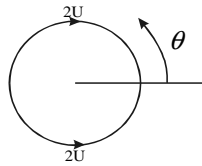
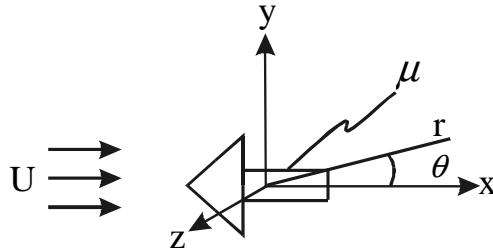


Illustration of the points where the flow reaches maximum speed around the circle.

$$\mathbf{3D:} \quad \phi = Ux + \frac{\mu}{4\pi} \frac{\cos \theta}{r^2} = Ur \cos \theta \left(1 + \frac{\mu}{4\pi r^3} \right)$$



The radial velocity is then

$$v_r = \frac{\partial \phi}{\partial r} = \cos \theta \left(U - \frac{\mu}{2\pi r^3} \right)$$

Setting the radial velocity $v_r = 0$ on $r = a$ we obtain $a = \sqrt[3]{\frac{\mu}{2\pi U}}$. This is the K.B.C. for a stationary sphere of radius a . Therefore, choosing

$$\mu = 2\pi U a^3$$

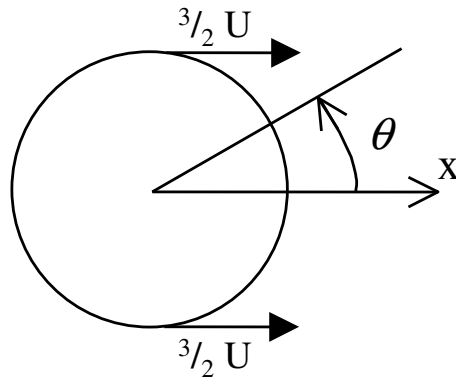
the potential

$$\phi = \cos \theta \left(Ur + \frac{\mu}{2\pi r} \right)$$

is **the** solution to ideal flow past a sphere of radius a .

- *Flow past a sphere* (U, a) .

$$\begin{aligned}\phi &= Ur \cos \theta \left(1 + \frac{a^3}{2r^3} \right) \\ v_\theta &= \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \sin \theta \left(1 + \frac{a^3}{2r^3} \right) \\ v_\theta|_{r=a} &= -\frac{3U}{2} \sin \theta \begin{cases} = 0 & \text{at } \theta = 0, \pi \\ = -\frac{3U}{2} & \text{at } \theta = \frac{\pi}{2} \end{cases}\end{aligned}$$



8. **2D corner flow** Velocity potential $\phi = r^\alpha \cos \alpha\theta$; Stream function $\psi = r^\alpha \sin \alpha\theta$

(a) $\nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi = 0$

(b)

$$u_r = \frac{\partial \phi}{\partial r} = \alpha r^{\alpha-1} \cos \alpha\theta$$

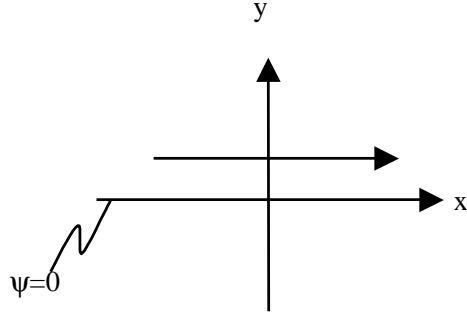
$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\alpha r^{\alpha-1} \sin \alpha\theta$$

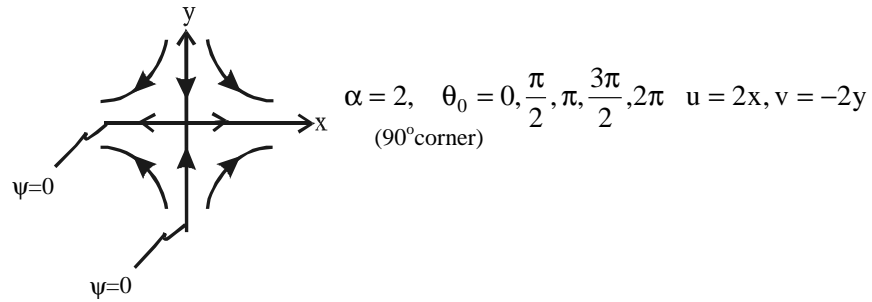
$$\therefore u_\theta = 0 \{ \text{ or } \psi = 0 \} \text{ on } \alpha\theta = n\pi, n = 0, \pm 1, \pm 2, \dots$$

i.e., on $\theta = \theta_0 = 0, \frac{\pi}{\alpha}, \frac{2\pi}{\alpha}, \dots (\theta_0 \leq 2\pi)$

i. **Interior corner flow** – stagnation point origin: $\alpha > 1$. For example,

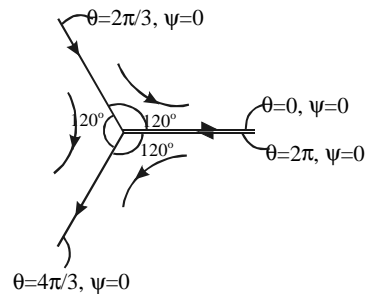
$$\alpha = 1, \theta_0 = 0, \pi, 2\pi, \quad u = 1, \quad v = 0$$





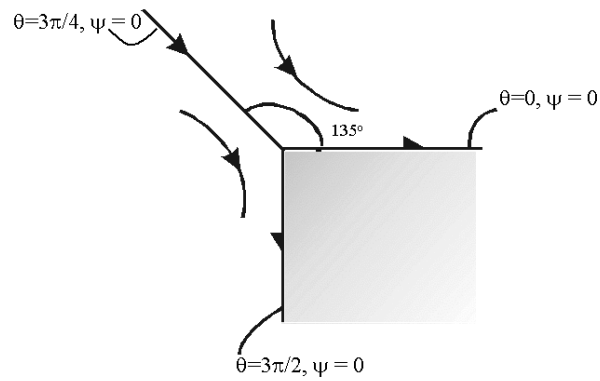
$$\alpha = 3/2, \quad \theta_0 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

(120° corner)



$$\alpha = 4/3, \quad \theta_0 = 0, \frac{3\pi}{4}, \frac{3\pi}{2}$$

(135° corner)



ii. **Exterior corner flow**, $|v| \rightarrow \infty$ at origin:

$$\alpha < 1$$

$$\theta_0 = 0, \frac{\pi}{\alpha} \text{ **only**}$$

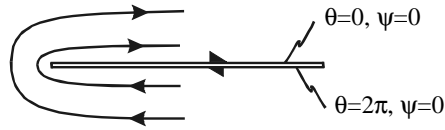
Since we need $\theta_0 \leq 2\pi$, we therefore require $\frac{\pi}{\alpha} \leq 2\pi$, i.e., $\alpha \geq 1/2$ only.

$$1/2 \leq \alpha < 1$$

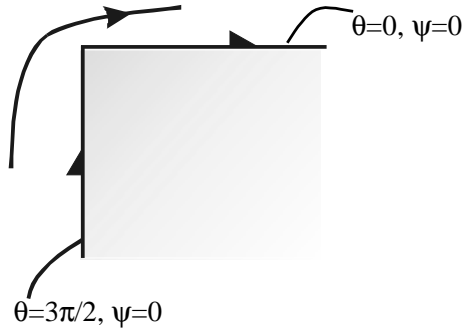
$$\theta_0 = 0, \frac{\pi}{\alpha}$$

For example,

$\alpha = 1/2, \theta_0 = 0, 2\pi$ ($1/2$ infinite plate, flow around a tip)







$\alpha = 2/3, \theta_0 = 0, \frac{3\pi}{2}$ (90° exterior corner)







Appendix A1: Summary of Simple Potential Flows

Cartesian Coordinate System

Flow	Streamlines	Potential $\phi(x, y, z)$	Stream function $\psi(x, y)$
Uniform flow		$U_{\infty} x + V_{\infty} y + W_{\infty} z$	$U_{\infty} y - V_{\infty} x$
2D Source/Sink (m) at (x_o, y_o)		$\frac{m}{2\pi} \ln((x - x_o)^2 + (y - y_o)^2)$	$\frac{m}{2\pi} \arctan(\frac{y - y_o}{x - x_o})$
3D Source/Sink (m) at (x_o, y_o, z_o)	NA	$-\frac{m}{4\pi} \frac{1}{\sqrt{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2}}$	NA
Vortex (Γ) at (x_o, y_o)		$\frac{\Gamma}{2\pi} \arctan(\frac{y - y_o}{x - x_o})$	$-\frac{\Gamma}{2\pi} \ln((x - x_o)^2 + (y - y_o)^2)$
2D Dipole (μ) at (x_o, y_o) at an angle α		$-\frac{\mu}{2\pi} \frac{(x - x_o) \cos \alpha + (y - y_o) \sin \alpha}{(x - x_o)^2 + (y - y_o)^2}$	$\frac{\mu}{2\pi} \frac{(y - y_o) \cos \alpha + (x - x_o) \sin \alpha}{(x - x_o)^2 + (y - y_o)^2}$
3D Dipole ($+x$) (μ) at (x_o, y_o, z_o)	NA	$-\frac{\mu}{4\pi} \frac{(x - x_o)}{((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2)^{3/2}}$	NA

Appendix A2: Summary of Simple Potential Flows

Cylindrical Coordinate System

Flow	Streamlines	Potential $\phi(r, \theta, z)$	Stream function $\psi(r, \theta)$
Uniform flow		$U_\infty r \cos \theta + V_\infty r \sin \theta + W_\infty z$	$U_\infty r \sin \theta - V_\infty r \cos \theta$
2D Source/Sink (m) at (x_o, y_o)		$\frac{m}{2\pi} \ln r$	$\frac{m}{2\pi} \theta$
3D Source/Sink (m) at (x_o, y_o, z_o)	NA	$-\frac{m}{4\pi r}$	NA
Vortex (Γ) at (x_o, y_o)		$\frac{\Gamma}{2\pi} \theta$	$-\frac{\Gamma}{2\pi} \ln r$
2D Dipole (μ) at (x_o, y_o) at an angle α		$-\frac{\mu}{2\pi} \frac{\cos \theta \cos \alpha + \sin \theta \sin \alpha}{r}$	$\frac{\mu}{2\pi} \frac{\sin \theta \cos \alpha + \cos \theta \sin \alpha}{r}$
3D Dipole ($+x$) (μ) at (x_o, y_o, z_o)	NA	$-\frac{\mu}{4\pi} \frac{\cos \theta}{r^2}$	NA

Appendix A3: Combination of Simple Potential Flows

Stream + Source = Rankine <i>Half</i> Body	(2D) (3D)	$\phi = U_\infty x + \frac{m}{2\pi} \ln r$ $x_s = -\frac{m}{2\pi U_\infty}$ $D = \frac{m}{U_\infty}$ $\phi = U_\infty x - \frac{m}{4\pi} \frac{1}{\sqrt{x^2+y^2+z^2}}$ $x_s = -\sqrt{\frac{m}{4\pi U_\infty}}$ $A = \frac{m}{U_\infty}$
Stream + Source + Sink = Rankine <i>Closed</i> Body	(2D) (3D)	$\phi = U_\infty x + \frac{m}{2\pi} [\ln((x+a)^2+y^2) - \ln((x-a)^2+y^2)]$ $\phi = U_\infty x + \frac{m}{4\pi} \left(\frac{1}{\sqrt{(x+a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2}} \right)$
Stream + Dipole = Circle (Sphere) $R = a$	(2D) (3D)	$\phi = U_\infty x + \frac{\mu x}{2\pi r^2}$ if $\mu = 2\pi a^2 U_\infty$ $\phi = U_\infty \cos \theta (r + \frac{a^2}{r})$ $\phi = U_\infty x + \frac{\mu \cos \theta}{4\pi r^2}$ if $\mu = 2\pi a^3 U_\infty$ $\phi = U_\infty \cos \theta (r + \frac{a^3}{2r^2})$
2D Corner Flow	(2D)	$\phi = Cr^\alpha \cos(\alpha\theta)$ $\psi = Cr^\alpha \sin(\alpha\theta)$ $\theta_0 = 0, \frac{n\pi}{\alpha}$

Appendix B: Far Field Behavior of Simple Potential Flows

Far field behavior $r \gg 1$		ϕ	$\vec{v} = \nabla\phi$
Source	(2D)	$\sim \ln r$	$\sim \frac{1}{r}$
	(3D)	$\sim \frac{1}{r}$	$\sim \frac{1}{r^2}$
Dipole	(2D)	$\sim \frac{1}{r}$	$\sim \frac{1}{r^2}$
	(3D)	$\sim \frac{1}{r^2}$	$\sim \frac{1}{r^3}$
Vortex	(2D)	~ 1	$\sim \frac{1}{r}$