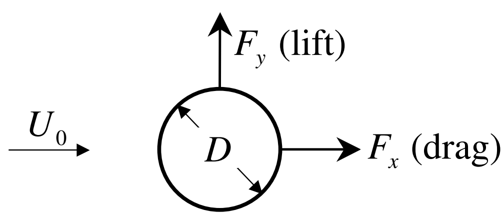
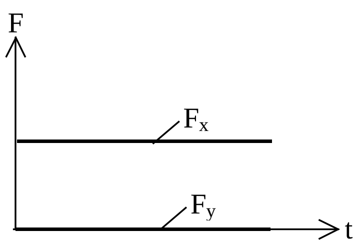
13.021 - Marine Hydrodynamics
Lecture 16

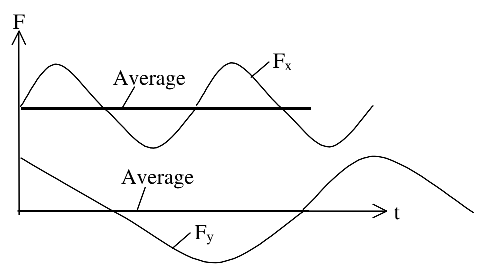
4.1.4 **Vortex Shedding and Vortex Induced Vibrations**
Consider a steady flow U_o over a bluff body with diameter D.



We would **expect** the average forces to be:

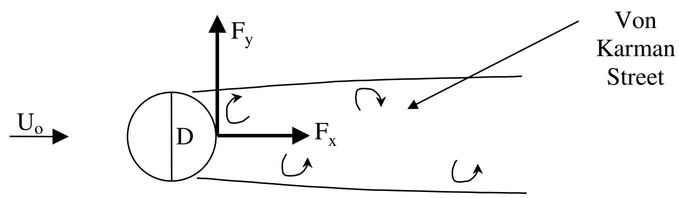


However, the **measured** oscillatory forces are:



- The **measured** drag F_x is found to oscillate about a **non-zero mean** value with frequency $2f$.
- The **measured** lift F_y is found to oscillate about a **zero mean** value with frequency f.
- $f = \omega/2\pi$ is the frequency of vortex shedding or Strouhal frequency.

Reason: Flow separation leads to vortex shedding. The vortices are shed in a staggered array, within an unsteady non-symmetric wake called **von Karman Street**. The frequency of vortex shedding is the Strouhal frequency and is a function of U_o, D, and ν.

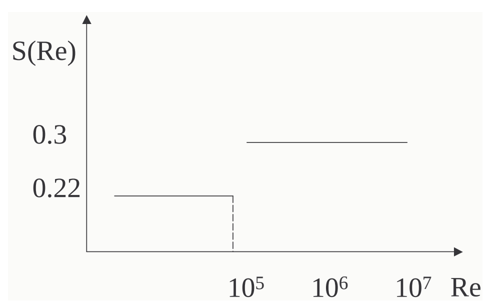


i) **Strouhal Number** We define the (dimensionless) Strouhal number $\boxed{S \equiv \frac{\overbrace{f}^{\text{Strouhal frequency}} D}{U_0}}$.

The Strouhal number S has a regime dependence on the R_e number $S = S(R_e)$.

For a **cylinder**:

- Laminar flow $S \sim 0.22$
- Turbulent flow $S \sim 0.3$



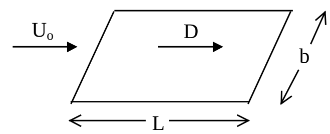
ii) **Drag and Lift** The drag and lift coefficients C_D and C_L are functions of the *correlation length.*

For '∞' correlation length:

- If the cylinder is fixed, $C_L \sim O(1)$ comparable to C_D.
- If the cylinder is free to move, as the Strouhal frequency f_S approaches one of the cylinder's natural frequencies f_n, 'lock-in' occurs. Therefore, if one natural frequency is close to the Strouhal Frequency $f_n \sim f_S$, we have large amplitude motions $\Rightarrow$**Vortex Induced Vibration** (VIV).

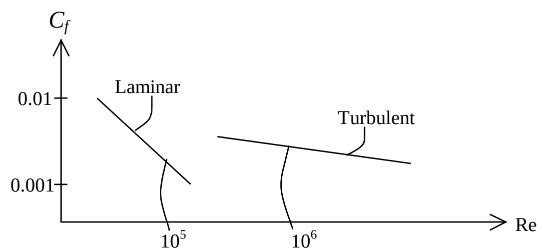
4.2 Drag on a Very Streamlined Body

$$
\begin{aligned}
R_{e_L} &\equiv \frac{UL}{\nu} \\
C_f &\equiv \frac{D}{\frac{1}{2}\rho U^2 \underbrace{(Lb)}_{\substack{S=\text{wetted area} \\ \text{one side of plate}}}}
\end{aligned}
$$



$$\boxed{C_f = C_f(R_{e_L}, L/b)}$$

Unlike a bluff body, C_f is a strong function of R_{e_L} since D is proportional to ν $\left(\tau = \nu \frac{\partial u}{\partial y}\right)$. See an example of C_f versus R_{e_L} for a flat plate in the figure below.



Skin friction coefficient as a function of the R_e for a flat plate

- R_{e_L} depends on plate smoothness, ambient turbulence, ...
- In general, C_f's are much smaller than C_D's ($C_f/C_D \sim O(0.1)$ to $O(0.01)$). Therefore, designing streamlined bodies allows minimal separation and smaller form drag at the expense of friction drag.
- In general, for streamlined bodies $C_{\text{Total Drag}}$ is a combination of $C_D\,(R_e)$ and $C_f\,(R_{e_L})$, and the total drag is $D = \frac{1}{2}\rho U^2 \left(C_D \underset{\text{frontal area}}{S} + C_f \underset{\text{wetted area}}{A_w} \right)$, where C_D has a regime dependence on R_e and C_f is a continuous function R_{e_L}.

4.3 Known Solutions of the Navier-Stokes Equations

4.3.1 Boundary Value Problem

- Navier-Stokes':

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} + \frac{1}{\rho}\vec{f}$$

- Conservation of mass:

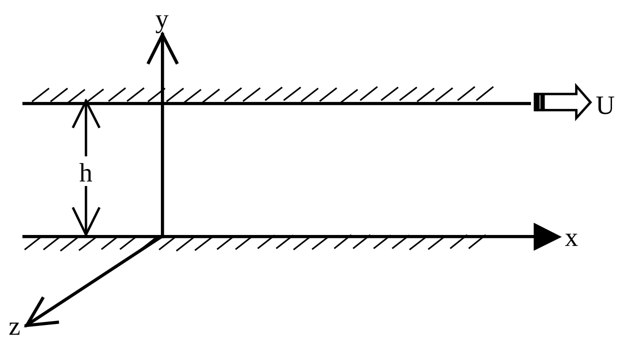
$$\nabla \cdot \vec{v} = 0$$

- Boundary conditions on solid boundaries "no-slip":

$$\vec{v} = \vec{U}$$

Equations very difficult to solve, analytic solution only for a few very special cases (usually when $\vec{v} \cdot \nabla \vec{v} = 0 \ldots$)

4.3.2 **Steady Laminar Flow Between 2 Long Parallel Plates: Plane Couette Flow**



Steady, viscous, incompressible flow between two infinite plates. The flow is driven by a pressure gradient in x and/or motion of the upper plate with velocity U parallel to the x-axis. Neglect gravity.

Assumptions		**Governing Equations**		**Boundary Conditions**
i. **Steady Flow:**	$\frac{\partial}{\partial t} = 0$	**Continuity:**	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$	$\vec{v} = (0,0,0)$ on $y = 0$
ii. $(x,z) >> h$**:**	$\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial z} = 0$	**NS:**	$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$	$\vec{v} = (0,0,0)$ on $y = h$
iii. **Pressure:**	independent of z			

Continuity

$$\underbrace{\frac{\partial u}{\partial x}}_{=0,} + \frac{\partial v}{\partial y} + \underbrace{\frac{\partial w}{\partial z}}_{\text{from assumption ii}} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = v(x,z) \underset{\substack{\uparrow \\ \text{BC: } v(x,0,z)=0}}{\Rightarrow} v = 0 \tag{1}$$

Momentum x

$$\underbrace{\frac{\partial u}{\partial t}}_{=0,\ \text{i}} + u\underbrace{\frac{\partial u}{\partial x}}_{=0,\ \text{ii}} + \underbrace{v}_{=0,\ (1)}\frac{\partial u}{\partial y} + w\underbrace{\frac{\partial u}{\partial z}}_{=0,\ \text{ii}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{=0,\ \text{ii}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{=0,\ \text{ii}}\right) \Rightarrow$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2}$$

Momentum y

$$\underbrace{\frac{\partial v}{\partial t}}_{=0,\ \text{i}} + \vec{v}\cdot\nabla\underbrace{v}_{=0,\ (1)} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2\underbrace{v}_{=0,\ (1)} \Rightarrow$$

$$\frac{\partial p}{\partial y} = 0 \underset{\substack{\uparrow \\ \text{assumption iii}}}{\Rightarrow} p = p(x) \text{ and } \frac{\partial p}{\partial x} = \frac{dp}{dx} \tag{3}$$

Momentum z

$$\underbrace{\frac{\partial w}{\partial t}}_{=0,\ \text{i}} + u\underbrace{\frac{\partial w}{\partial x}}_{=0,\ \text{ii}} + \underbrace{v}_{=0,\ (1)}\frac{\partial u}{\partial y} + w\underbrace{\frac{\partial w}{\partial z}}_{=0,\ \text{ii}} = -\frac{1}{\rho}\underbrace{\frac{\partial p}{\partial z}}_{=0,\ \text{iii}} + \nu\left(\underbrace{\frac{\partial^2 w}{\partial x^2}}_{=0,\ \text{ii}} + \frac{\partial^2 w}{\partial y^2} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{=0,\ \text{ii}}\right) \Rightarrow$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow w = ay + b \underset{\substack{\uparrow \\ w(x,0,z)=0 \\ w(x,h,z)=0}}{\Rightarrow} w = 0 \tag{4}$$

From Equations (1), (4)

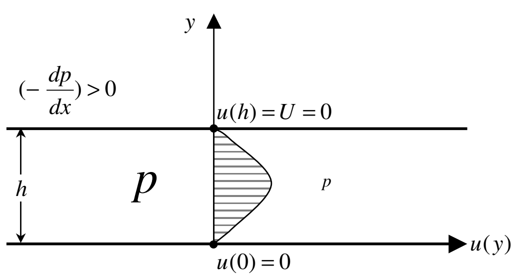
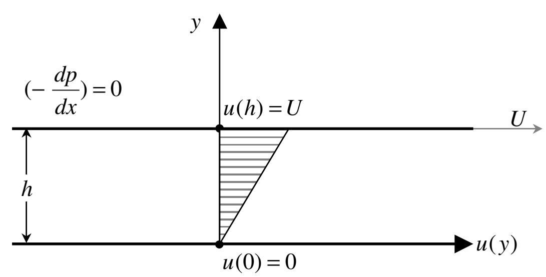
$$\vec{v} = (u, 0, 0).\ \text{Also} \underset{\substack{\uparrow \\ \text{assumption ii}}}{\Rightarrow} u = u(y) \text{ and } \frac{\partial u}{\partial y} = \frac{du}{dy} \tag{5}$$

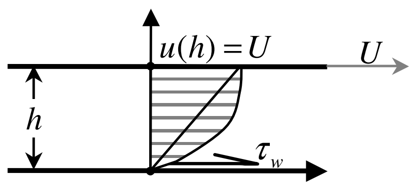
From Equations (2), (3, and (5)

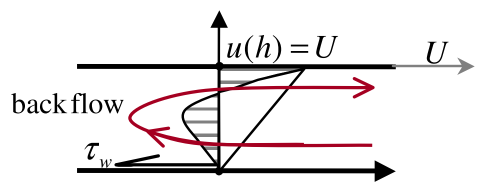
$$\frac{d^2 u}{dy^2} = \frac{1}{\rho\nu}\frac{d^2 p}{dx^2} \underset{\substack{\uparrow \\ \mu=\rho\nu}}{\Rightarrow} u = -\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)y^2 + C_1 y + C_2 \underset{\substack{\uparrow \\ u(x,0,z)=0 \\ w(x,h,z)=U}}{\Rightarrow} \boxed{u = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)(h-y)y + U\frac{y}{h}}$$

• Special cases for Couette flow

$u(y) = \frac{1}{2\mu}(h-y)y(-\frac{dp}{dx}) + U\frac{y}{h}$, where $(-\frac{dp}{dx}) = \frac{P_x - P_{x+L}}{L}$

I. $U = 0,\ \left(-\frac{dp}{dx}\right) > 0$	**II.** $U \neq 0,\ \left(-\frac{dp}{dx}\right) = 0$
 Parabolic profile	 Linear profile
•Velocity $u(y) = \frac{1}{2\mu}(h-y)y(-\frac{dp}{dx})$	$u(y) = U\frac{y}{h}$
•Max velocity $u_{max} = u(h/2) = \frac{h^2}{8\mu}(-\frac{dp}{dx})$	$u_{max} = U$
•Volume flow rate $Q = \int_0^h u(y)dy = \frac{h^3}{8\mu}(-\frac{dp}{dx})$	$Q = \frac{h}{2}U$
•Average velocity $\bar{u} = \frac{Q}{h} = \frac{h^2}{6\mu}(-\frac{dp}{dx})$	$\bar{u} = \frac{U}{2}$
•Viscous stress on bottom plate (skin friction) $\tau_w = \mu \left.\frac{du}{dy}\right\|_{y=0} = \frac{h}{2}\left(-\frac{dp}{dx}\right) > 0$	$\tau_w = \mu \left.\frac{du}{dy}\right\|_{y=0} = \mu\frac{U}{h}$

<table>
<tr><td colspan="2" align="center">III. $U \neq 0, \left(-\frac{dp}{dx}\right) \neq 0$</td></tr>
<tr><td align="center">

$U > 0, (-\frac{dp}{dx}) > 0, G > 0$</td><td align="center">

$U > 0, (-\frac{dp}{dx}) < 0, G < 0$</td></tr>
<tr><td align="center">$\left(-\frac{dp}{dx}\right) > 0$</td><td align="center">$\left(-\frac{dp}{dx}\right) < 0$</td></tr>
<tr><td colspan="2">•Viscous stress on bottom plate (skin friction)

$$\tau_w = \frac{h}{2}\left(-\frac{dp}{dx}\right) + \mu\frac{U}{h}$$

$$\tau_w \begin{smallmatrix}<\\=\\>\end{smallmatrix} 0 \text{ when } \left(-\frac{dp}{dx}\right) \begin{smallmatrix}<\\=\\>\end{smallmatrix} -\frac{2\mu U}{h^2}, \text{ in which case the flow is } \begin{cases}\text{attached}\\ \text{insipient}\\ \text{separated}\end{cases}$$</td></tr>
</table>

For the general case of $U \neq 0$ and $\left(-\frac{dp}{dx}\right) \neq 0$,

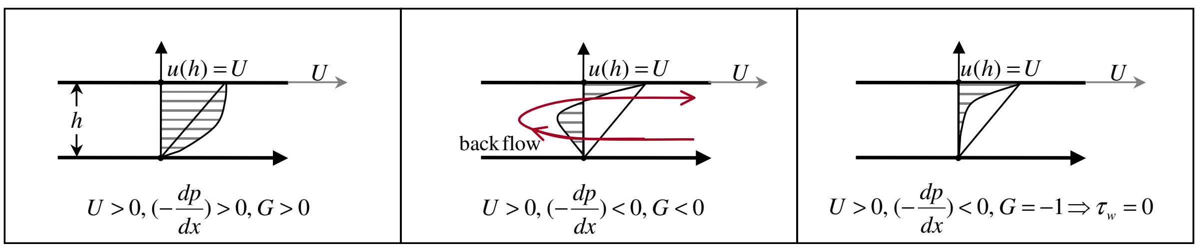
$$\tau_w = \frac{h}{2}\left(-\frac{dp}{dx}\right) + \mu\frac{U}{h}$$

We define a **Dimensionless Pressure Gradient** G

$$\boxed{G \equiv \frac{h^2}{2\mu U}\left(-\frac{dp}{dx}\right)}$$

such that

- $G > 0$ denotes a **favorable** pressure gradient
- $G < 0$ denotes an **adverse** pressure gradient
- $G = -1$ denotes an **incipient** flow
- $G < -1$ denotes a **separated or back-flow**

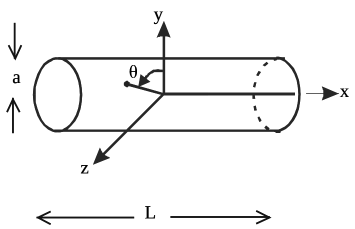


Lessons learned in § 4.3.2:

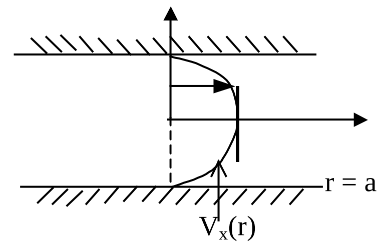
1. Reviewed how to simplify the Navier-Stokes equations.
2. Obtained one solution to the Navier-Stokes equations.
3. Realized that once the Navier-Stokes are solved we know **everything**.

In the next paragraph we are going to study one more solution to the Navier-Stokes equation, in polar coordinates.

4.3.3 **Steady Laminar Flow in a Pipe: Poiseuille Flow**



Steady, laminar pipe flow.
($r^2 = y^2 + z^2$, $\vec{v} = (v_x, v_r, v_\theta)$)



KBC: $v_x(a) = 0$ (no slip) and
$\frac{dv_x}{dr}(0) = 0$ (symmetry).

Assumptions		Governing Equations	Boundary Conditions
i. **Steady Flow:**	$\frac{\partial}{\partial t} = 0$	**Continuity:** $\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_x}{\partial x} = 0$	$v_x(r=a) = 0$ no-slip
ii. $(x, z) >> h$:	$\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial \theta} = 0$	**NS:** *In polar coordinates (see SAH pp.74)*	$\frac{dv_x}{dr}\|_{r=0} = 0$ symmetry
	$\Rightarrow \vec{v} = \vec{v}(r)$		
iii. **Pressure:**	independent of θ		

Following a procedure similar to that for plane Couette flow (*left as an exercise*) we can show that

$$v_r = v_\theta = 0,\ v_x = v_x(r),\ p = p(x), \quad \text{and}\ \frac{1}{\rho}\frac{dp}{dx} = \nu \underbrace{\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_x}{dr}\right)\right)}_{\substack{r \text{ component of } \nabla^2 \\ \text{in cylindrical coordinates}}}$$

After applying the boundary conditions we find:

$$\boxed{v_x(r) = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)\left(a^2 - r^2\right)}$$

Therefore the volume flow rate is given by

$$Q = \int_0^{2\pi} d\theta \int_0^a r dr v_x(r) = \frac{\pi}{8\mu}a^4\big(-\frac{dp}{dx}\big)$$

and the skin friction evaluates to

$$\tau_w = \tau_x(-r) = -\tau_{xy} = -\mu\,\left(\frac{\partial v_r}{\partial x} + \frac{\partial v_x}{\partial r}\right)\bigg|_{r=a} = -\mu\,\frac{\partial v_x}{\partial r}\bigg|_{r=a} \Rightarrow \tau_w = \frac{a}{2}\big(-\frac{dp}{dx}\big)$$

4.4 Boundary Layer Growth Over an Infinite Flat Plate for Unsteady Flow

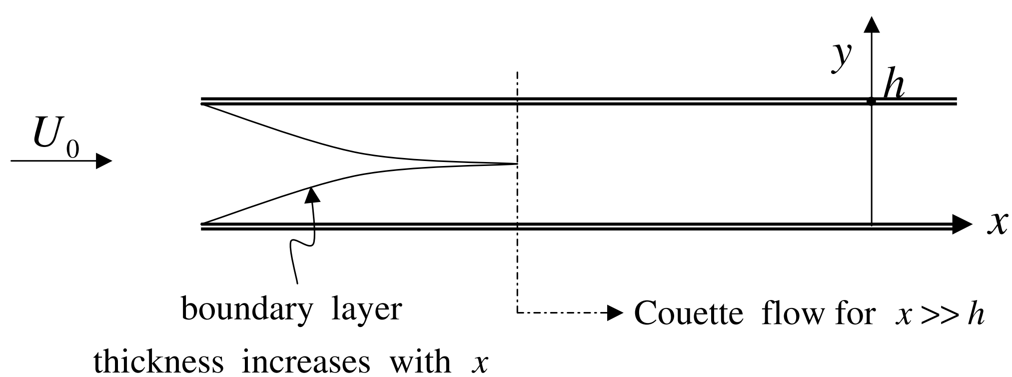
Boundary layer thickness is related to the area where the viscosity and vorticity effects are diffused.

For a flow over an infinite flat plate, the boundary layer thickness increases unless it is constrained in the y direction and/or by time (unsteady flow).

1. **Steady flow, constrained in y**

   For a **steady** flow past a flat plate, the boundary layer thickness increases with x. If the flow is constrained in y, eventually the viscous effects are diffused along the entire cross section and the flow becomes invariant in the streamwise direction.

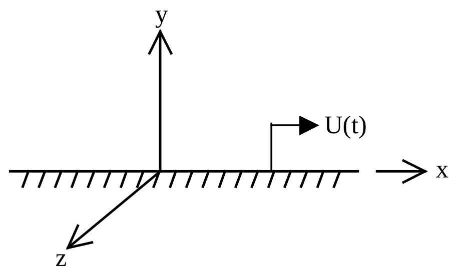
   In paragraphs 4.3.2 and 4.3.3, we studied two cases of steady laminar viscous flows, where the viscous effects had diffused along the entire cross section.



Steady $\begin{pmatrix} \text{Couette} \\ \text{Poiseuille} \end{pmatrix}$ flow, we *assumed* that viscous effects diffused through entire $\begin{pmatrix} h \\ a \end{pmatrix}$.

2. **Unsteady flow, unconstrained in y**

Consider the simplest example of an infinite plate in unsteady motion.



Assumptions $\nabla p = 0$, $\frac{\partial \vec{v}}{\partial x} = \frac{\partial \vec{v}}{\partial z} = 0 \Rightarrow \vec{v} = \vec{v}\,(y,t)$

Can show $v = w = 0$ and $u = u(y,t)$.

Finally, from u momentum (Navier-Stokes in x) we obtain

$$\frac{\partial u}{\partial t} + u\underbrace{\frac{\partial u}{\partial x}}_{=0} + \underbrace{v}_{=0}\frac{\partial u}{\partial y} + w\underbrace{\frac{\partial u}{\partial z}}_{=0} = -\frac{1}{\rho}\underbrace{\frac{\partial p}{\partial x}}_{=0} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{=0} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{=0}\right) \Rightarrow$$

$$\boxed{\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}} \quad \text{`}\underbrace{\text{momentum}}_{\substack{\text{velocity}\\\text{(heat)}}}\text{' diffusion equation} \tag{6}$$

Equation (6) is:

⋆ first order PDE in time → requires 1 *Initial Condition*

⋆ second order PDE in y → requires 2 *Boundary Conditions*

- $u(y,t) = U(t)$ at $y = 0$, for $t > 0$
- $u(y,t) \to 0$ as $y \to \infty$

From Equation (6), we observe that the flow over a moving flat plate is due to viscous dissipation only.

4.5.1 **Sinusoidally Oscillating Plate**

i. **Evaluation of the Velocity Profile for Stokes Boundary Layer**

The flow over an oscillating flat plate is referred to as 'Stokes Boundary Layer'.

Recall that $e^{i\alpha} = \cos\alpha + i\sin\alpha$ where α is real.

Assume that the plate is oscillating with $U(t) = U_o \cos\omega t = \text{Real}\,\{U_o e^{i\omega t}\}$. From linear theory, it is *known* that the fluid velocity must have the form

$$u\,(y,t) = \text{Real}\left\{f\,(y)\,e^{i\omega t}\right\}, \tag{7}$$

where f(y) is the unknown complex (magnitude & phase) amplitude of oscillation.

To obtain an expression for $f(y)$, simply substitute (7) in (6). This leads to:

$$i\omega f = \nu \frac{d^2 f}{dy^2} \tag{8}$$

Equation (8) is a 2^{nd} order **ODE** for $f(y)$. The general solution is

$$f\,(y) = C_1 e^{(1+i)\left(\sqrt{\omega/2\nu}\right)y} + C_2 e^{-(1+i)\left(\sqrt{\omega/2\nu}\right)y} \tag{9}$$

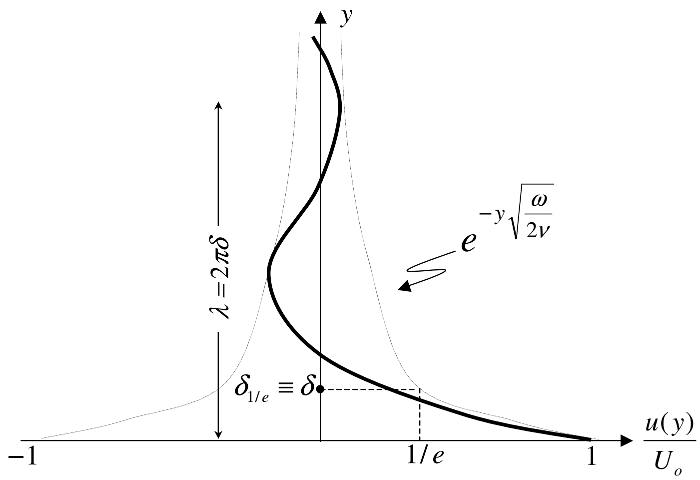
The velocity profile is obtained from Equations (7), (9) after we apply the Boundary Conditions.

$$\left.\begin{array}{l} u(y,t) \text{ must be bounded as } y \to \infty \Leftrightarrow C_1 = 0 \\ u(y=0,t) = U(t) \Leftrightarrow f(y=0) = U_o \Leftrightarrow C_2 = U_o \end{array}\right\} \boxed{u\,(y,t) = U_o(e^{-y\sqrt{\frac{\omega}{2\nu}}})\cos\left(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\right)}$$

Stokes Boundary Layer

ii. **Some Calculations for the Stokes Boundary Layer**

Once the velocity profile is evaluated, we know everything about the flow.



Stokes Boundary Layer. Velocity ratio $\frac{u(y)}{U_o}$ as a function of the distance from the plate y.

Observe:

$$\frac{u(y,t)}{U_o} = \underbrace{(e^{-y\sqrt{\frac{\omega}{2\nu}}})}_{\text{Exponentially decaying envelope}} \underbrace{\cos\big(-y\sqrt{\frac{\omega}{2\nu}} + \omega t\big)}_{\text{Oscillating component}} \tag{10}$$

SBL thickness

The ratio $\frac{u}{U_o}$ is composed of an exponentially decaying part $\rightarrow$ thickness of SBL decays exponentially with y. We *define* various parameters that can be used as measures of the SBL thickness:

- We define $\delta_{1/e}$ as the distance y from the plate where $\frac{u(\delta_{1/e})}{U_o} = \frac{1}{e}$. Substituting into (10), we find that $\delta \equiv \delta_{1/e} = \sqrt{\frac{2\nu}{\omega}}$
- The oscillating component has wave length $\lambda = 2\pi\sqrt{\frac{2\nu}{\omega}} = 2\pi\delta$. At λ, $\frac{u(\lambda)}{U_o} \cong 0.002$.
- We define $\delta_{1\%}$ as the distance y from the plate, where $\frac{u(\delta_{1\%})}{U_o} = 1\%$. Substituting into (10), we find that $\delta_{1\%} = -\ln(\frac{u(\delta_{1\%})}{U_o})\sqrt{\frac{2\nu}{\omega}} \cong 4.6\delta$.

Numerical examples:

For oscillating plate in water ($\nu = 10^{-6}$m^2/s= 1mm^2/s) we have

$$\underbrace{\delta_{1\%}}_{\text{in mm}} = \frac{4.6}{\sqrt{\pi}}\sqrt{T} \cong 2.6\sqrt{\underbrace{T}_{\text{in sec}}}$$

$T = \frac{2\pi}{\omega}$	$\delta_{1\%}$
1s	3mm
10s	≤1cm

Excursion length and SBL

The plate undergoes a motion of amplitude A.

$$X = A\sin(\omega t) \Rightarrow U = \dot{X} = \underbrace{A\omega}_{U_o}\cos(\omega t) \Rightarrow \omega = \frac{U_o}{A}$$

Comparing the SBL thickness $\sim \delta$ with A, we find

$$\frac{\delta}{A} \sim \frac{\sqrt{\nu/\omega}}{A} \underset{\omega=\frac{U_o}{A}}{\overset{\uparrow}{=}} \frac{\sqrt{\nu A/U_o}}{A} = \sqrt{\frac{\nu}{U_o A}} \sim \frac{1}{\sqrt{R_{e_A}}}$$

Skin friction

The skin friction on the plate is given by

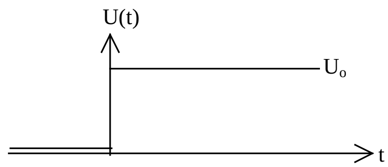
$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \ldots = \mu U_o\sqrt{\frac{\omega}{2\nu}}\left(\sin\omega t - \cos\omega t\right)$$

The maximum skin friction on the wall is

$$|\tau_w|_{max} = \mu U_o\sqrt{\frac{\omega}{\nu}}$$

and occurs at $\omega t = \frac{3\pi}{4}, \frac{7\pi}{4}, \cdots$

4.5.2 **Impulsively Started Plate**



Recall Equation (6) that describes the the flow $u(y,t)$ over an infinite flat plate undergoing unsteady motion.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

For an impulsively started plate, the Boundary Conditions are:

$$\left.\begin{array}{l} u(o,t) = U_o \\ u(\infty,t) = 0 \end{array}\right\} \text{ for } t > 0, \text{i.e. } u(y,0) = 0$$

Notice that the problem stated by Equation (6) with the above Boundary Conditions has no explicit time scale. In this case it is standard procedure to (a) use **Dimensional Analysis** to find the similarity parameters of the problem, and (b) look for solution in terms of the similarity parameters:

$$u = \mathrm{f}\,(U_o, y, t, \nu) \underset{\substack{\uparrow \\ \text{DA}}}{\Rightarrow} \frac{u}{U_o} = f\Big(\underbrace{\frac{y}{2\sqrt{\nu t}}}_{\substack{\equiv \eta \\ \text{similarity parameter}}}\Big) \Rightarrow \boxed{\frac{u}{U_o} = f(\eta)} \qquad \text{Self similar solution}$$

The velocity profile is thus given by*:

$$\frac{u}{U_o} = \underbrace{\operatorname{erfc}(\eta)}_{\substack{\text{Complementary} \\ \text{error function}}} = 1 - \operatorname{erf}(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\alpha^2} d\alpha$$

$\star$ Hints on obtaining the solution:

$$\left.\begin{aligned} &\eta = \tfrac{y}{2\sqrt{\nu t}} \\ &\tfrac{\partial}{\partial t} = \tfrac{\partial \eta}{\partial t}\tfrac{\partial}{\partial \eta} = -\tfrac{y}{4t\sqrt{\nu t}}\tfrac{\partial}{\partial \eta} \\ &\tfrac{\partial^2}{\partial y^2} = \left(\tfrac{\partial \eta}{\partial y}\right)^2 \tfrac{\partial^2}{\partial \eta^2} = \tfrac{1}{4\nu t}\tfrac{\partial^2}{\partial \eta^2} \end{aligned}\right\} \xrightarrow{\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}} \underbrace{-\eta\frac{d(u/U_o)}{d\eta} = \frac{d^2(u/U_o)}{d\eta^2}}_{2^{nd}\text{ order ODE}} \longrightarrow \ldots$$

Boundary layer thickness
In the same manner as for the SBL, we define various parameters that can be used to measure the boundary layer thickness:

- $\delta \equiv 2\sqrt{\nu t}$. At $y = \delta \longrightarrow \frac{u(\delta)}{U_o} \cong 0.16$.
- $\delta_{1\%} \cong 1.82\delta$.

Excursion length and boundary layer thickness
At time t, the plate has travelled a distance $L = U_o t \rightarrow t = \frac{L}{U_o}$.
Comparing the boundary layer thickness $\sim \delta$ with L, we find

$$\frac{\delta}{L} \sim \frac{\sqrt{\nu t}}{L} = \frac{\sqrt{\nu L/U_o}}{L} = \sqrt{\frac{\nu}{U_o L}} \sim \frac{1}{\sqrt{R_{e_L}}}$$

Skin friction
The skin friction on the plate is given by

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \ldots = -\mu\frac{U_o}{\sqrt{\pi \nu t}}$$