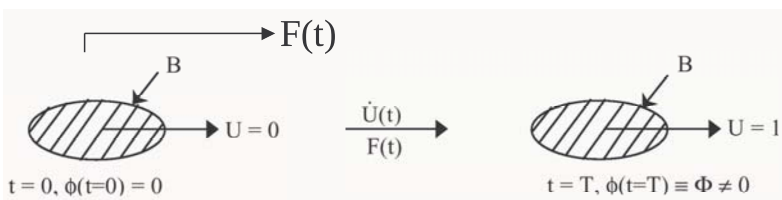
13.021 - Marine Hydrodynamics
Lecture 14

3.20 Some Properties of Added-Mass Coefficients

1. $m_{ij} = \rho \cdot$[function of geometry only]

   F, M = [linear function of m_{ij}] $\times$ [function of $\underbrace{\text{instantaneous}}_{\text{not of motion history}}$ $U, \dot{U}, \Omega$]

2. Relationship to fluid momentum.



where we define Φ to denote the velocity potential that corresponds to unit velocity $U = 1$. In this case the velocity potential ϕ for an arbitrary velocity U is $\phi = U\Phi$.
The linear momentum $\vec{L}$ in the fluid is given by

$$\vec{L} = \iiint_V \rho \vec{v} dV = \iiint_V \rho \nabla \phi dV \underset{\substack{\uparrow \\ \text{Green's} \\ \text{theorem}}}{=} \int_B + \underbrace{\int_{\infty}}_{\phi \to 0 \text{ at } \infty} \rho \phi \hat{n} dS$$

$$L_x (t = T) = \iint_B \rho U \Phi n_x dS = U \iiint_B \rho \Phi n_x dS$$

The force exerted on the fluid from the body is $-F(t) = -(-m_A \dot{U}) = m_A \dot{U}$.

$$\int_0^T dt\,[-F(t)] = \int_0^T m_A \dot{U} dt = \underbrace{m_A\, U]_0^T}_{m_A U} \overset{\substack{\text{Newton's Law}\\ \downarrow}}{=} L_x(t=T) - L_x(t=0) = U \iint_B \rho \Phi n_x dS$$

Therefore, m_A = total fluid momentum for a body moving at $U = 1$ (regardless of how we get there from rest) = fluid momentum per unit velocity of body.

K.B.C. $\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n} = (U, 0, 0) \cdot \hat{n} = U n_x,\ \frac{\partial \phi}{\partial n} = U n_x \Rightarrow \frac{\partial U \Phi}{\partial n} = U n_x \Rightarrow \boxed{\dfrac{\partial \Phi}{\partial n} = n_x}$

$$\therefore\ m_A = \rho \iint_B \Phi \frac{\partial \Phi}{\partial n} dS$$

For general 6 DOF:

$$\underbrace{m_{ji}}_{\substack{j-\text{force/moment}\\ i-\text{direction of motion}}} = \rho \iint_B \underbrace{\Phi_i}_{\substack{\text{potential due to body}\\ \text{moving with } U_i=1}} n_j dS = \rho \iint_B \Phi_i \frac{\partial \Phi_j}{\partial n} dS =$$ j fluid momentum due to i body motion

3. Symmetry of added mass matrix $m_{ij} = m_{ji}$.

$$m_{ji} = \rho \iint_B \Phi_i \left(\frac{\partial \Phi_j}{\partial n} \right) dS = \rho \iint_B \Phi_i \left(\nabla \Phi_j \cdot \hat{n} \right) dS \underset{\substack{\uparrow\\ \text{Divergence}\\ \text{Theorem}}}{=} \rho \iiint_V \nabla \cdot \left(\Phi_i \nabla \Phi_j \right) dV$$

$$= \rho \iiint_V \left(\nabla \Phi_i \cdot \nabla \Phi_j + \Phi_i \underbrace{\nabla^2 \Phi_j}_{=0} \right) dV$$

Therefore,

$$m_{ji} = \rho \iiint_V \nabla \Phi_i \cdot \nabla \Phi_j dV = m_{ij}$$

4. Relationship to the kinetic energy of the fluid. For a general 6 DoF body motion $U_i = (U_1, U_2, \ldots, U_6)$,

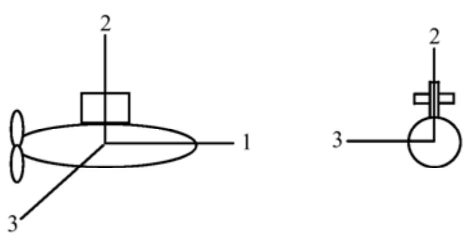
$$\phi = \underbrace{U_i \Phi_i}_{\sum \text{ notation}} \; ; \Phi_i = \text{ potential for } U_i = 1$$

$$\begin{aligned} K.E. &= \frac{1}{2}\rho \iiint\limits_V \nabla\phi \cdot \nabla\phi dV = \tfrac{1}{2}\rho \iiint\limits_V U_i \nabla\Phi_i \cdot U_j \nabla\Phi_j dV \\ &= \frac{1}{2}\rho U_i U_j \iiint\limits_V \nabla\Phi_i \cdot \nabla\Phi_j dV = \tfrac{1}{2} m_{ij} U_i U_j \end{aligned}$$

K.E. depends only on m_{ij} and <u>instantaneous</u> U_i.

5. Symmetry simplifies m_{ij}. From 36 $\underset{\text{symmetry}}{\longrightarrow}$ 21 $\rightarrow$ '?'. Choose such coordinate system that some $m_{ij} = 0$ by symmetry.

Example 1 Port-starboard symmetry.

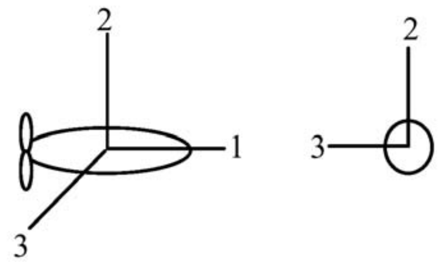


$m_{\underset{\text{force}}{3}\underset{\text{motion}}{1}} = 0$ by symmetry $\therefore$ $m_{13} = 0$; (NOT OBVIOUS!)

$$m_{ij} = \begin{array}{c|c} \begin{bmatrix} m_{11} & m_{12} & 0 & 0 & 0 & m_{16} \\ & m_{22} & 0 & 0 & 0 & m_{26} \\ & & m_{33} & m_{34} & m_{35} & 0 \\ & & & m_{44} & m_{45} & 0 \\ & & & & m_{55} & 0 \\ & & & & & m_{66} \end{bmatrix} & \begin{matrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{matrix} \\ \begin{matrix} U_1 & U_2 & U_3 & \Omega_1 & \Omega_2 & \Omega_3 \end{matrix} & \end{array}$$

12 independent coefficients

Example 2 Rotational or axi-symmetry with respect to x_1 axis.



$$m_{ij} = \begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ & m_{22} & 0 & 0 & 0 & m_{35} \\ & & m_{22} & 0 & m_{35} & 0 \\ & & & 0 & 0 & 0 \\ & & & & m_{55} & 0 \\ & & & & & m_{55} \end{bmatrix} \quad \begin{array}{l} \text{where } m_{22} = m_{33}, m_{55} = m_{66} \\ \text{and } m_{26} = m_{35},\ \text{so 4 different coefficients} \end{array}$$

Exercise How about 3 planes of symmetry (e.g. a cuboid); a cube; a sphere?? Work out the details.

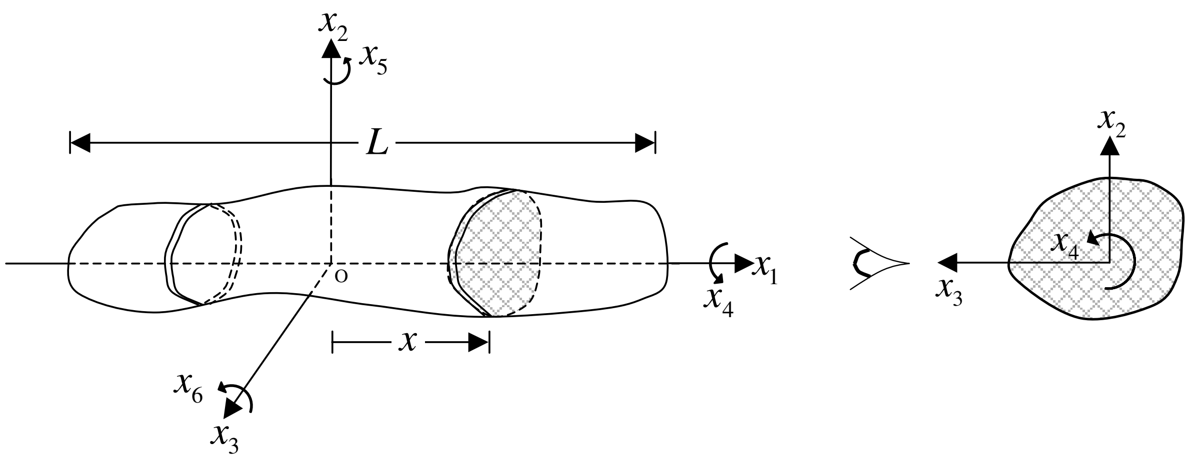
3.21 Slender Body Approximation

Definitions

(a) Slender Body = a body whose characteristic length in the longitudinal direction is considerably larger than the body's characteristic length in the other two directions.

(b) m_{ij} = the 3D added mass coefficient in the i^{th} direction due to a unit acceleration in the j^{th} direction. The subscripts i, j run from 1 to 6.

(c) M_{kl} = the 2D added mass coefficient in the k^{th} direction due to a unit acceleration in the l^{th} direction. The subscripts k, l take the values 2,3 and 4.



Goal To estimate the added mass coefficients m_{ij} for a 3D *slender body.*

Idea Estimate m_{ij} of a *slender* 3D body using the 2D sectional added mass coefficients (strip-wise M_{kl}). In particular, for simple shapes like long cylinders, we will use *known* 2D coefficients to find unknown 3D coefficients.

$$\underset{3D}{m_{ij}} = \sum [\underset{2D}{M_{kl}(x)} \text{ contributions}]$$

Discussion If the 1-axis is the longitudinal axis of the slender body, then the 3D added mass coefficients m_{ij} are calculated by summing the added mass coefficients of all the thin slices which are perpendicular to the 1-axis, M_{kl}. **This means that forces in 1-direction cannot be obtained by slender body theory**.

Procedure In order to calculate the 3D added mass coefficients m_{ij} we need to:

1. Determine the 2D acceleration of each crossection for a unit acceleration in the i^{th} direction,
2. Multiply the 2D acceleration by the *appropriate* 2D added mass coefficient to get the force on that section in the j^{th} direction, and
3. Integrate these forces over the length of the body.

Examples

- Sway force due to sway acceleration
Assume a unit sway acceleration $\dot{u}_3 = 1$ and all other $u_j, \dot{u}_j = 0$, with $j = 1, 2, 4, 5, 6$. It then follows from the expressions for the generalized forces and moments (Lecture 12, JNN §4.13) that the sway force on the body is given by

$$f_3 = -m_{33}\dot{u}_3 = -m_{33} \Leftrightarrow m_{33} = -f_3 = -\int_L F_3(x)dx$$

A unit 3 acceleration in 3D results to a unit acceleration in the 3 direction of each 2D 'slice' ($\dot{U}_3 = \dot{u}_3 = 1$). The hydrodynamic force on each slice is then given by

$$F_3(x) = -M_{33}(x)\dot{U}_3 = -M_{33}(x)$$

Putting everything together, we obtain

$$m_{33} = -\int_L -M_{33}(x)dx = \int_L M_{33}(x)dx$$

- Sway force due to yaw acceleration
Assume a unit yaw acceleration $\dot{u}_5 = 1$ and all other $u_j, \dot{u}_j = 0$, with $j = 1, 2, 3, 4, 6$. It then follows from the expressions for the generalized forces and moments that the sway force on the body is given by

$$f_3 = -m_{35}\dot{u}_5 = -m_{35} \Leftrightarrow m_{35} = -f_3 = -\int_L F_3(x)dx$$

For each 2D 'slice', a distance x from the origin, a unit 5 acceleration in 3D, results to a unit acceleration in the -3 direction times the moment arm x ($\dot{U}_3 = -x\dot{u}_5 = -x$). The hydrodynamic force on each slice is then given by

$$F_3(x) = -M_{33}(x)\dot{U}_3 = xM_{33}(x)$$

Putting everything together, we obtain

$$m_{35} = -\int_L x M_{33}(x) dx$$

- Yaw moment due to yaw acceleration
  Assume a unit yaw acceleration $\dot{u}_5 = 1$ and all other $u_j, \dot{u}_j = 0$, with $j = 1, 2, 3, 4, 6$. It then follows from the expressions for the generalized forces and moments that the yaw force on the body is given by

  $$f_5 = -m_{55}\dot{u}_5 = -m_{55} \Leftrightarrow m_{55} = -f_5 = -\int_L F_5(x) dx$$

  For each 2D 'slice', a distance x from the origin, a unit 5 acceleration in 3D, results to a unit acceleration in the -3 direction times the moment arm x ($\dot{U}_3 = -x\dot{u}_5 = -x$). The hydrodynamic force on each slice is then given by

  $$F_3(x) = -M_{33}(x)\dot{U}_3 = x M_{33}(x)$$

  However, each force $F_3(x)$ produces a negative moment at the origin about the 5 axis

  $$F_5(x) = -x F_3(x)$$

  Putting everything together, we obtain

  $$m_{55} = \int_L x^2 M_{33}(x) dx$$

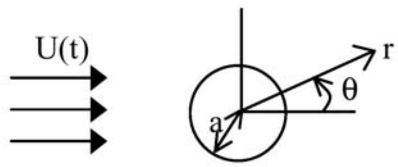
In the same manner we can estimate the remaining added mass coefficients m_{ij} - **noting that added mass coefficients related to the 1-axis cannot be obtained by slender body theory**.

In summary, the 3D added mass coefficients are shown in the following table. The empty boxes may be filled in by symmetry.

$$\left[\begin{array}{c|c|c|c|c}
m_{22} = \int\limits_L M_{22} dx & m_{23} = \int\limits_L M_{23} dx & m_{24} = \int\limits_L M_{24} dx & m_{25} = \int\limits_L -x M_{23} dx & m_{26} = \int\limits_L x M_{22} dx \\
\hline
 & m_{33} = \int\limits_L M_{33} dx & m_{34} = \int\limits_L M_{34} dx & m_{35} = \int\limits_L -x M_{33} dx & m_{36} = \int\limits_L x M_{32} dx \\
\hline
 & & m_{44} = \int\limits_L M_{44} dx & m_{45} = \int\limits_L -x M_{34} dx & m_{46} = \int\limits_L x M_{24} dx \\
\hline
 & & & m_{55} = \int\limits_L x^2 M_{33} dx & m_{56} = \int\limits_L -x^2 M_{32} dx \\
\hline
 & & & & m_{66} = \int\limits_L x^2 M_{22} dx
\end{array}\right]$$

3.22 Buoyancy Effects Due to Accelerating Flow

Example Force on a stationary sphere in a fluid that is accelerated against it.



$$\phi\left(r,\theta,t\right)=U\left(t\right)\left(r+\underbrace{\frac{a^3}{2r^2}}_{\text{dipole for sphere}}\right)\cos\theta$$

$$\left.\frac{\partial\phi}{\partial t}\right|_{r=a}=\dot{U}\frac{3a}{2}\cos\theta$$

$$\left.\nabla\phi\right|_{r=a}=\left(0,-\frac{3}{2}U\sin\theta,0\right)$$

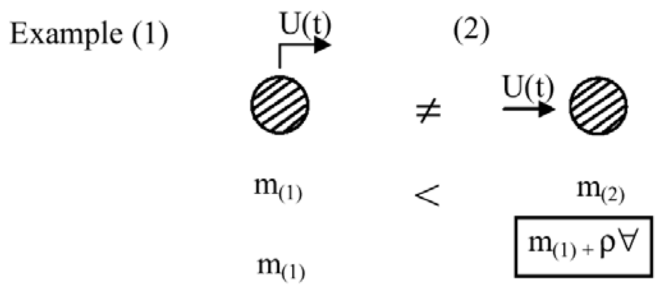
$$\frac{1}{2}\left.\left|\nabla\phi\right|^2\right|_{r=a}=\frac{9}{8}U^2\sin^2\theta$$

Then,

$$F_x=\left(-\rho\right)\left(2\pi r^2\right)\int_0^\pi d\theta\sin\theta\left(-\cos\theta\right)\left[\dot{U}\frac{3a}{2}\cos\theta+\frac{9}{8}U^2\sin^2\theta\right]$$

$$=\dot{U}3\pi\rho a^3\underbrace{\int_0^\pi d\theta\sin\theta\cos^2\theta}_{=2/3}+\rho U^2\frac{9\pi}{4}a^2\underbrace{\int_0^\pi d\theta\cos\theta\sin^3\theta}_{=0}$$

$$F_x=\dot{U}\underbrace{\rho\left(2\pi a^3\right)}_{\substack{\frac{4}{3}\pi a^3\rho+\frac{2}{3}\pi a^3\rho\\ \Downarrow\\ =\rho\forall}}$$



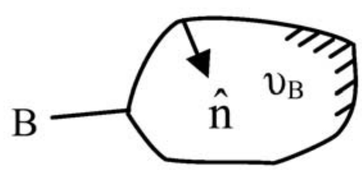
Part of F_x is due to the **pressure gradient** which must be present to cause the fluid to accelerate:

x-momentum, noting $U = U(t)$: $\dfrac{\partial U}{\partial t} + U \underbrace{\dfrac{\partial U}{\partial x}}_{0} + \underbrace{v\dfrac{\partial U}{\partial y}}_{0} + \underbrace{w\dfrac{\partial U}{\partial z}}_{0} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$ (ignore gravity)

$$\frac{dp}{dx} = -\rho \dot{U} \quad \text{for uniform (1D) accelerated flow}$$

Force on the body due to the pressure field

$$\vec{F} = \iint_B p\hat{n}dS = -\iiint_{V_B} \nabla p dV; \quad F_x = -\iiint_{V_B} \frac{\partial p}{\partial x} dV = \rho \forall \dot{U}$$



'Buoyancy' force due to pressure gradient $= \rho\forall\dot{U}$

Analogue: Buoyancy force due to hydrostatic pressure gradient. Gravitational acceleration g $\leftrightarrow$ $\dot{U}$= fluid acceleration.

$$\begin{aligned}&p_s = -\rho g y\\ &\nabla p_s = -\rho g \hat{j} \rightarrow \vec{F}_s = -\rho g \forall \hat{j} \quad \text{Archimedes principle}\end{aligned}$$

Summary: Total force on a fixed sphere in an accelerated flow

$$F_x = \dot{U}\Bigg(\underbrace{\rho\forall}_{\text{Buoyancy}} + \underbrace{m_{(1)}}_{\substack{\text{added mass}\\ \frac{1}{2}\rho\forall}}\Bigg) = \dot{U}\tfrac{3}{2}\rho\forall = 3\dot{U}m_{(1)}$$

In general, for any body in an accelerated flow:

$$F_x = F_{\text{buoyany}} + \dot{U}m_{(1)},$$

where $m_{(1)}$ is the added mass in still water (from now on, m)

$$F_x = -\dot{U}m \text{ for body acceleration with } \dot{U}$$

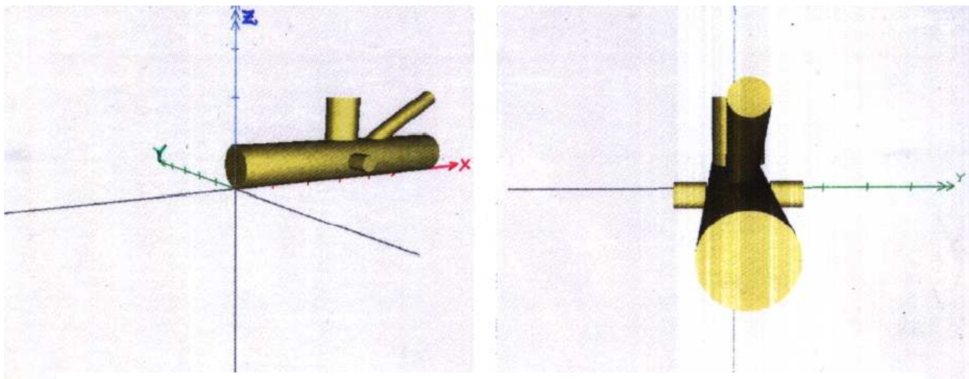
Added mass coefficient

$$c_m = \frac{m}{\rho\forall}$$

in the presence of accelerated flow $C_m = 1 + c_m$

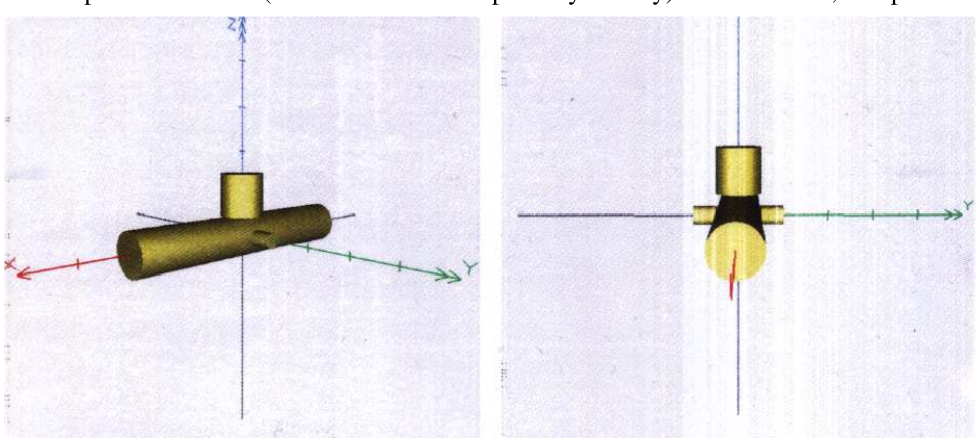
Appendix A: More examples on symmetry of added mass tensor

•Symmetry with respect to Y (= “X-Z” plane symmetry) **12** non-zero, independent coefficients



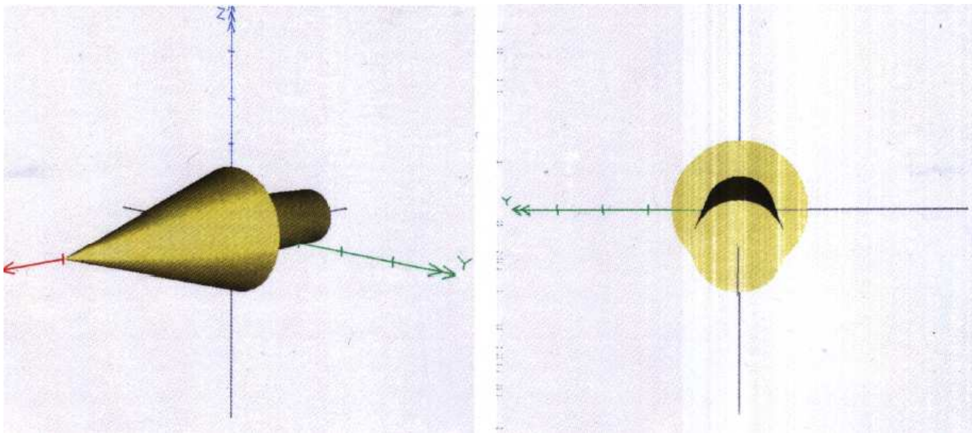
$$\frac{m_{ij}}{\rho} = \begin{vmatrix} 1.008 & 0.000 & -0.179 & 0.000 & 1.887 & 0.000 \\ 0.000 & 5.682 & 0.000 & -3.625 & 0.000 & 17.696 \\ -0.179 & 0.000 & 5.304 & 0.000 & -16.616 & 0.000 \\ 0.000 & -3.625 & 0.000 & 3.051 & 0.000 & -11.734 \\ 1.887 & 0.000 & -16.616 & 0.000 & 63.471 & 0.000 \\ 0.000 & 17.696 & 0.000 & -11.734 & 0.000 & 64.761 \end{vmatrix}$$

•Symmetry with respect to X and Y (= “Y-Z” and “X-Z” plane symmetry) **7** non-zero, independent coefficients



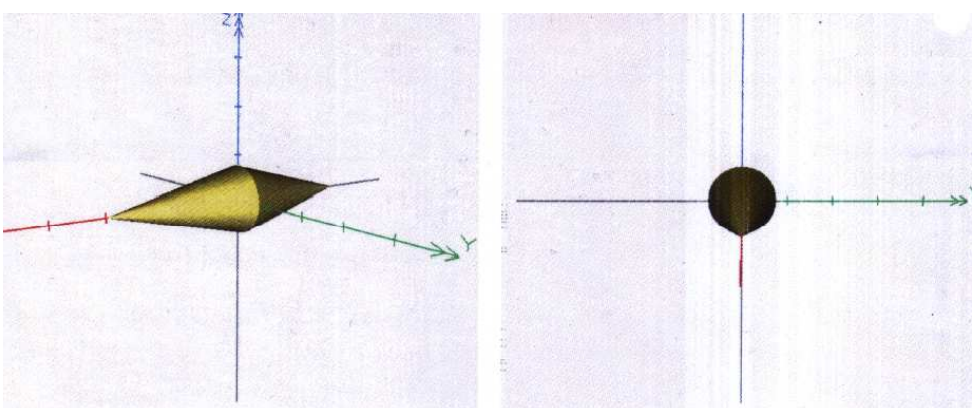
$$\frac{m_{ij}}{\rho} = \begin{vmatrix} 1.173 & 0.000 & 0.000 & 0.000 & 0.785 & 0.000 \\ 0.000 & 5.531 & 0.000 & -0.785 & 0.000 & 0.000 \\ 0.000 & 0.000 & 5.100 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.785 & 0.000 & 0.897 & 0.000 & 0.000 \\ 0.785 & 0.000 & 0.000 & 0.000 & 9.307 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 8.555 \end{vmatrix}$$

•Axisymmetric with respect to X-axis **4** non-zero, independent coefficients



$$\frac{m_{ij}}{\rho} = \begin{vmatrix} 4.418 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 19.439 & 0.000 & 0.000 & 0.000 & 1.473 \\ 0.000 & 0.000 & 19.439 & 0.000 & -1.473 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.473 & 0.000 & 29.394 & 0.000 \\ 0.000 & 1.473 & 0.000 & 0.000 & 0.000 & 29.394 \end{vmatrix}$$

•Axisymmetric with respect to X axis and X (="Y-Z" plane symmetry) **3** non-zero, independent coefficients



$$\frac{m_{ij}}{\rho} = \begin{vmatrix} 0.884 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 5.301 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 5.301 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 3.181 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 3.181 \end{vmatrix}$$