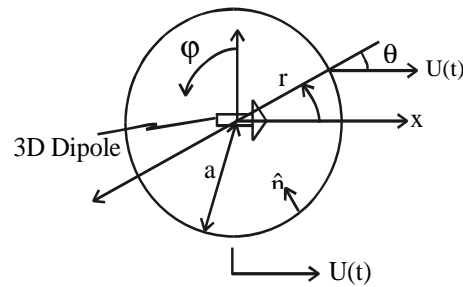


## 13.021 - Marine Hydrodynamics Lecture 13

### 3.18 Unsteady Motion - Added Mass

**D'Alembert:** ideal, irrotational, unbounded, steady.

**Example** Force on a sphere accelerating ( $U = U(t)$ , unsteady) in an unbounded fluid that is at rest at infinity.



K.B.C on sphere:  $\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = U(t) \cos \theta$

Solution: Simply a 3D dipole (no stream)

$$\phi = -U(t) \frac{a^3}{2r^2} \cos \theta$$

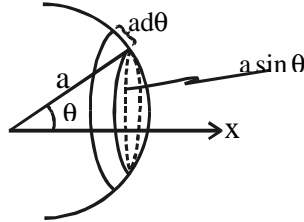
Check:  $\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = U(t) \cos \theta$

Hydrodynamic force:

$$F_x = -\rho \iint_B \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS$$

On  $r = a$ ,

$$\begin{aligned} \left. \frac{\partial \phi}{\partial t} \right|_{r=a} &= -\dot{U} \frac{a^3}{2r^2} \cos \theta \Big|_{r=a} = -\frac{1}{2} \dot{U} a \cos \theta \\ \nabla \phi|_{r=a} &= \left( \frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) = \left( U \cos \theta, \frac{1}{2} U \sin \theta, 0 \right) \\ |\nabla \phi|^2|_{r=a} &= U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta; \hat{n} = -\hat{e}_r, n_x = -\cos \theta \\ \iint_B dS &= \int_0^\pi (a d\theta) (2\pi a \sin \theta) \end{aligned}$$



Finally,

$$\begin{aligned}
F_x &= (-\rho) 2\pi a^2 \int_0^\pi d\theta (\sin \theta) \underbrace{\left( -\cos \theta \right)}_{n_x} \left[ \underbrace{-\frac{1}{2} \dot{U} a \cos \theta}_{\frac{\partial \phi}{\partial t}} + \frac{1}{2} \underbrace{\left( U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta \right)}_{|\nabla \phi|^2} \right] \\
F_x &= -\dot{U}(\rho a^3) \pi \underbrace{\int_0^\pi d\theta \sin \theta \cos^2 \theta}_{2/3} + (\rho U^2) \pi a^2 \underbrace{\int_0^\pi d\theta \sin \theta \cos \theta \left( \cos^2 \theta + \frac{1}{4} \sin^2 \theta \right)}_{= 0, \text{ D'Alembert revisited}} \\
\underbrace{F_x}_{\text{Hydrodynamic Force}} &= - \underbrace{\dot{U}(t)}_{\text{Acceleration}} \left[ \underbrace{\rho}_{\text{Fluid Density}} \underbrace{\frac{2}{3} \pi a^3}_{\text{Volume} = 1/2 V_{\text{sphere}}} \right]
\end{aligned}$$

Thus the **Hydrodynamic Force** on a sphere of diameter  $a$  moving with velocity  $U(t)$  in an unbounded fluid of density  $\rho$  is given by

$$F_x = -\dot{U}(t) \left[ \rho \frac{2}{3} \pi a^3 \right]$$

Comments:

- If  $\dot{U} = 0 \rightarrow F_x = 0$ , i.e., steady translation  $\rightarrow$  no force (D'Alembert's Condition ok).
- $F_x \propto \dot{U}$  with a  $(-)$  sign, i.e., the fluid tends to 'resist' the acceleration.
- $[\dots]$  has the units of (**fluid**) mass  $\equiv m_a$
- Equation of Motion for a body of mass  $M$  that moves with velocity  $U$ :

$$\underbrace{M}_{\text{Body mass}} \dot{U} = \Sigma F = \underbrace{F_H}_{\text{Hydrodynamic force } \iint_S p \hat{n} dS} + \underbrace{F_B}_{\text{All other forces on body}} = \left( -\dot{U} \underbrace{m_a}_{\text{Fluid mass}} \right) + F_B \Leftrightarrow$$

$$(M + m_a) \dot{U} = F_B$$

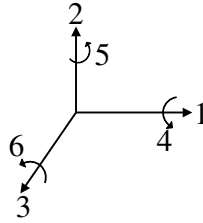
i.e., the presence of fluid around the body acts as an **added** or **virtual** mass to the body.

## 3.19 General 6 Degrees of Freedom Motions

### 3.19.1 Notation Review

(3D)  $U_1, U_2, U_3$ : Translational velocities

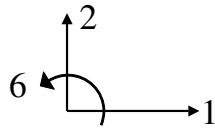
$U_4 \equiv \Omega_1, U_5 \equiv \Omega_2, U_6 \equiv \Omega_3$ : Rotational velocities



(2D)  $U_1, U_2$ : Translational velocities

$U_6 \equiv \Omega_3$ : Rotational velocity

$U_3 = U_4 = U_5 = 0$



### 3.19.2 Added Mass Tensor (matrix)

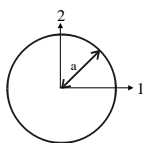
$$m_{ij} ; i, j = 1, 2, 3, 4, 5, 6$$

$m_{ij}$ : associated with force on body in  $i$  direction due to unit acceleration in  $j$  direction. For example, for a sphere:

$$m_{11} = m_{22} = m_{33} = \frac{1}{2}\rho V = (m_A) \quad \text{all other } m_{ij} = 0$$

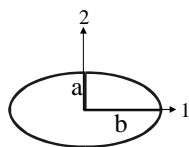
### 3.19.3 Added Masses of Simple 2D Geometries

- Circle



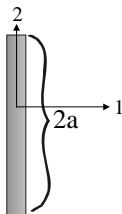
$$m_{11} = m_{22} = \rho \nabla = \rho \pi a^2$$

- Ellipse



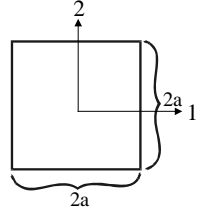
$$m_{11} = \rho \pi a^2, m_{22} = \rho \pi b^2$$

- Plate



$$m_{11} = \rho \pi a^2, m_{22} = 0$$

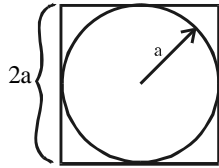
- Square



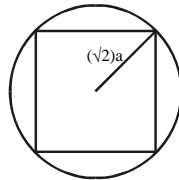
$$m_{11} = m_{22} \approx 4.754\rho a^2$$

A reasonable approximation to estimate the added mass of a 2D body is to use the displaced mass ( $\rho V$ ) of an ‘equivalent cylinder’ of the same lateral dimension or one that ‘rounds off’ the body. For example, consider a square and approximate with an

- (a) inscribed circle:  $m_A = \rho\pi a^2 = 3.14\rho a^2$ .



- (b) circumscribed circle:  $m_A = \rho\pi (\sqrt{2}a)^2 = 6.28\rho a^2$ .

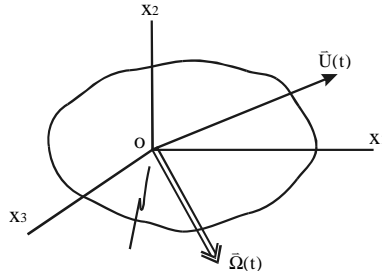


Arithmetic mean of (a) + (b)  $\approx 4.71\rho a^2$ .

### 3.19.4 Generalized Forces and Moments

In this paragraph we are looking at the most general case where forces and moments are induced on rigid body moving with 6 DoF motions, in an unbounded fluid that is at rest at infinity.

**Body fixed reference frame**, i.e.,  $OX_1X_2X_3$  is **fixed** on the body.



$$\vec{U}(t) = (U_1, U_2, U_3), \text{ translational velocity}$$

$$\vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6), \text{ rotational velocity with respect to } O$$

Consider a body with a 6 DoF motion  $(\vec{U}, \vec{\Omega})$ , and a fixed reference frame  $OX_1X_2X_3$ . Then the hydrodynamic forces and moments with respect to  $O$  are given by the following relations (JNN §4.13)

- Forces

$$F_j = -\dot{U}_{\underset{1.}{i}} m_{ji} - E_{jkl} U_{\underset{2.}{i}} \Omega_k m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6$$

$$\text{and} \quad j, k, l = 1, 2, 3$$

- Moments

$$M_j = -\dot{U}_{\underset{3.}{i}} m_{j+3,i} - E_{jkl} U_{\underset{2.}{i}} \Omega_k m_{l+3,i} - E_{jkl} U_{\underset{3.}{k}} U_{\underset{3.}{i}} m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6$$

$$\text{and} \quad j, k, l = 1, 2, 3$$

Einstein's  $\Sigma$  notation applies.

$$E_{jkl} = \text{'alternating tensor'} = \begin{cases} 0 & \text{if any } j, k, l \text{ are equal} \\ 1 & \text{if } j, k, l \text{ are in cyclic order, i.e.,} \\ & (1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2) \\ -1 & \text{if } j, k, l \text{ are not in cyclic order i.e.,} \\ & (1, 3, 2), (2, 1, 3), (3, 2, 1) \end{cases}$$

Note:

- (a) if  $\Omega_k \equiv 0$ ,  $F_j = -\dot{U}_i m_{ji}$  (as expected by definition of  $m_{ij}$ ).  
Also if  $\dot{U}_i \equiv 0$ , then  $F_j = 0$  for any  $U_i$ , no force in steady translation.
- (b)  $B_l \sim U_i m_{li}$  'added momentum' due to rotation of axes.  
Then all the terms marked as 2. are proportional to  $\sim \vec{\Omega} \times \vec{B}$  where  $\vec{B}$  is linear momentum (momentum from  $i$  coordinate into new  $x_j$  direction).
- (c) If  $\Omega_k \equiv 0$ :  $M_j = -\dot{U}_i m_{j+3,i} m_{ij} - \underbrace{E_{jkl} U_k U_i m_{li}}_{\text{even with } \dot{U}=0, M_j \neq 0 \text{ due to this term}}.$

**Moment** on a body due to pure steady translation – 'Munk' moment.



### 3.19.5 Example Generalized motions, forces and moments.

A certain body has non-zero added mass coefficients only on the diagonal, i.e.  $m_{ij} = \delta_{ij}$ . For a body motion given by  $U_1 = t$ ,  $U_2 = -t$ , and all other  $U_i$ ,  $\Omega_i = 0$ , the forces and moments on the body in terms of  $m_i$  are:

$$F_1 = \underline{\hspace{1cm}}, F_2 = \underline{\hspace{1cm}}, F_3 = \underline{\hspace{1cm}}, M_1 = \underline{\hspace{1cm}}, M_2 = \underline{\hspace{1cm}}, M_3 = \underline{\hspace{1cm}}$$

**Solution:**

$$m_{ij} = \delta_{ij}$$

$$U_1 = t \quad U_2 = -t \quad U_i = 0 \quad i = 3, 4, 5, 6 \quad \Omega_k = 0 \quad k = 1, 2, 3$$

$$\dot{U}_1 = 1 \quad \dot{U}_2 = -1 \quad \dot{U}_i = 0 \quad i = 3, 4, 5, 6$$

Use the relations from (JNN §4.13):

$$\begin{aligned} F_j &= -\dot{U}_i m_{ij} - E_{jkl} U_i \Omega_k m_{il} \xrightarrow{\Omega_k=0} \\ F_j &= -\dot{U}_i m_{ij} \end{aligned}$$

$$\begin{aligned} M_j &= -\dot{U}_i m_{i(j+3)} - E_{jkl} U_i \Omega_k m_{i(l+3)} - E_{jkl} U_k U_i m_{li} \xrightarrow{\Omega_k=0} \\ M_j &= -\dot{U}_i m_{i(j+3)} - E_{jkl} U_k U_i m_{li} \end{aligned}$$

where  $i = 1, 2, 3, 4, 5, 6$  and  $j, k, l = 1, 2, 3$

For  $F_1, F_2, F_3$  use the previous relationship for  $F_j$  with  $j = 1, 2, 3$  respectively:

$$\begin{aligned} F_1 &= -\underbrace{\dot{U}_1}_{=1} m_{11} - \underbrace{\dot{U}_2}_{=0} m_{21} - \underbrace{\dot{U}_3}_{=0} m_{31} - \underbrace{\dot{U}_4}_{=0} m_{41} - \underbrace{\dot{U}_5}_{=0} m_{51} - \underbrace{\dot{U}_6}_{=0} m_{61} \rightarrow \boxed{F_1 = -m_{11}} \\ F_2 &\stackrel{\text{Check}}{=} -\underbrace{\dot{U}_2}_{=-1} m_{22} \rightarrow \boxed{F_2 = m_{22}} \\ F_3 &\stackrel{\text{Check}}{=} -\underbrace{\dot{U}_3}_{=0} m_{33} \rightarrow \boxed{F_3 = 0} \end{aligned}$$

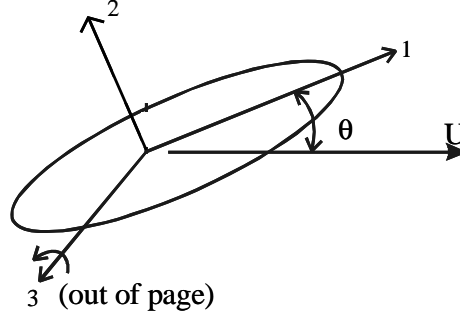
For  $M_1, M_2, M_3$  use the previous relationship for  $M_j$  with  $j = 1, 2, 3$  respectively:

$$\begin{aligned}
M_1 &= -\dot{U}_i m_{i(1+3)} - E_{1kl} U_k U_i m_{li} \\
&= -\dot{U}_i m_{i4} - E_{1kl} U_k U_i m_{li} \\
&= -\dot{U}_1 \underbrace{m_{14}}_{=0} - \dot{U}_2 \underbrace{m_{24}}_{=0} - \dot{U}_3 \underbrace{m_{34}}_{=0} - \underbrace{\dot{U}_4}_{=0} m_{44} - \dot{U}_5 \underbrace{m_{54}}_{=0} - \dot{U}_6 \underbrace{m_{64}}_{=0} \\
&\quad - E_{123} U_2 \left( U_1 \underbrace{m_{13}}_{=0} + U_2 \underbrace{m_{23}}_{=0} + \underbrace{U_3}_{=0} m_{33} + U_4 \underbrace{m_{43}}_{=0} + U_5 \underbrace{m_{53}}_{=0} + U_6 \underbrace{m_{63}}_{=0} \right) \\
&\quad - E_{132} \underbrace{U_3}_{=0} \left( U_1 \underbrace{m_{12}}_{=0} + \underbrace{U_2}_{=-1} m_{22} + U_3 \underbrace{m_{32}}_{=0} + U_4 \underbrace{m_{42}}_{=0} + U_5 \underbrace{m_{52}}_{=0} + U_6 \underbrace{m_{62}}_{=0} \right) \rightarrow \boxed{M_1 = 0}
\end{aligned}$$

$$\begin{aligned}
M_2 &= -\dot{U}_i m_{i5} - E_{2kl} U_k U_i m_{li} \\
&= \dot{U}_5 m_{55} - E_{231} U_3 U_i m_{1i} - E_{213} U_1 U_i m_{3i} \\
&= -E_{213} U_1 U_3 m_{33} \rightarrow \boxed{M_2 = 0}
\end{aligned}$$

$$\begin{aligned}
M_3 &= -\dot{U}_i m_{i6} - E_{3kl} U_k U_i m_{li} \\
&= \dot{U}_6 m_{66} - \underbrace{E_{312}}_{+1} U_1 U_i m_{2i} - \underbrace{E_{321}}_{-1} U_2 U_i m_{1i} \\
&= - \underbrace{U_1}_t \underbrace{U_2}_{-t} m_{22} + \underbrace{U_2}_{-t} \underbrace{U_1}_t m_{11} \rightarrow \boxed{M_3 = t^2(m_{22} - m_{11})}
\end{aligned}$$

### 3.19.6 Example Munk Moment on a 2D submarine in steady translation



$$\begin{aligned} U_1 &= U \cos \theta \\ U_2 &= -U \sin \theta \end{aligned}$$

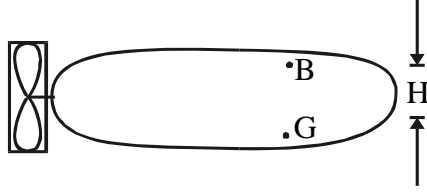
Consider steady translation motion:  $\dot{U} = 0; \Omega_k = 0$ . Then

$$M_3 = -E_{3kl}U_kU_lm_{li}$$

For a 2D body,  $m_{3i} = m_{i3} = 0$ , also  $U_3 = 0, i, k, l = 1, 2$ . This implies that:

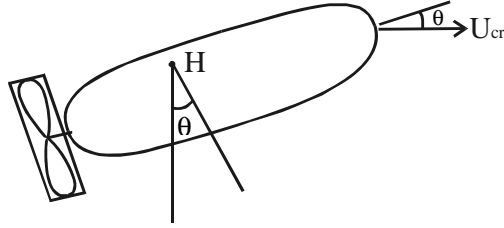
$$\begin{aligned} M_3 &= -\underbrace{E_{312}}_{=1}U_1(U_1m_{21} + U_2m_{22}) - \underbrace{E_{321}}_{=-1}U_2(U_1m_{11} + U_2m_{12}) \\ &= -U_1U_2(m_{22} - m_{11}) \\ &= U^2 \sin \theta \cos \theta \left( \underbrace{m_{22} - m_{11}}_{>0} \right) \end{aligned}$$

Therefore,  $M_3 > 0$  for  $0 < \theta < \pi/2$  ('Bow up'). Therefore, a submarine under forward motion is unstable in pitch (yaw). For example, a small bow-up tends to grow with time, and control surfaces are needed as shown in the following figure.



- Restoring moment  $\approx (\rho g \nabla H) \sin \theta$ .
- critical speed  $U_{cr}$  given by:

$$(\rho g \nabla) H \sin \theta \geq U_{cr}^2 \sin \theta \cos \theta (m_{22} - m_{11})$$



Usually  $m_{22} \gg m_{11}$ ,  $m_{22} \approx \rho \nabla$ . For small  $\theta$ ,  $\cos \theta \approx 1$ . So,  $U_{cr}^2 \leq gH$  or  $F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1$ . Otherwise, control fins are required.