13.021 – Marine Hydrodynamics
Lecture 18

4.7 – Turbulent Flow – Reynolds Stress

$$u = \overline{u} + u'$$

By definition: $\overline{u'} = \overline{u - \overline{u}} = \overline{u} - \overline{u} = 0$, also $\dfrac{\partial}{\partial x}\overline{u} = \overline{\dfrac{\partial u}{\partial x}}$ etc

Substitute into governing equations and take $\overline{(\ )}$:

$$\overline{\text{continuity}}: \quad 0 = \overline{\frac{\partial u_i}{\partial x_i}} = \overline{\frac{\partial \overline{u}_i}{\partial x_i}} + \underbrace{\overline{\frac{\partial u'_i}{\partial x_i}}}_{0} \qquad \therefore \frac{\partial \overline{u}_i}{\partial x_i} = 0$$

$$\text{but} \quad \frac{\partial u_i}{\partial x_i} = 0 = \underbrace{\frac{\partial \overline{u}_i}{\partial x_i}}_{0 \ \textit{just shown}} + \frac{\partial u'_i}{\partial x_i} \qquad \therefore \frac{\partial u'_i}{\partial x_i} = 0$$

Momentum Equation: $\dfrac{\partial u_i}{\partial t} + u_j \dfrac{\partial u_i}{\partial x_j} = \dfrac{1}{\rho}\dfrac{\partial \tau_{ij}}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$

$u_i = \overline{u}_i + u_i'$ & $\overline{(\ )}$:

$$\overline{\frac{\partial u_i}{\partial t}} = \frac{\partial \overline{u}_i}{\partial t} + \underbrace{\overline{\frac{\partial u_i'}{\partial t}}}_{0} \ ; \ \textit{similarly} \begin{cases} \overline{\nu \nabla^2 u_i} = \nu \nabla^2 \overline{u_i} \\ \overline{\dfrac{\partial p}{\partial x_i}} = \dfrac{\partial}{\partial x_i}\left(\overline{\overline{p} + p'}\right) = \dfrac{\partial \overline{p}}{\partial x_i} \quad etc \end{cases}$$

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\left(\overline{u}_j + u_j'\right)\frac{\partial}{\partial x_j}\left(\overline{u}_i + u_i'\right)} = \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \underbrace{\overline{u_j' \frac{\partial \overline{u}_i}{\partial x_j}}}_{0} + \underbrace{\overline{\overline{u}_j \frac{\partial u_i'}{\partial x_j}}}_{0} + \overline{u_j' \frac{\partial}{\partial x_j} u_i'}$$

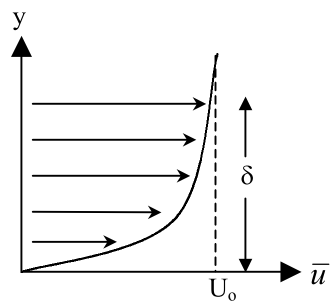
From continuity: $\overline{u_j' \dfrac{\partial}{\partial x_j} u_i'} = \dfrac{\partial}{\partial x_j}\overline{u_j' u_i'} - \overline{u'_i \underbrace{\dfrac{\partial u_j'}{\partial x_j}}_{0 \ \text{from continuity}}}$

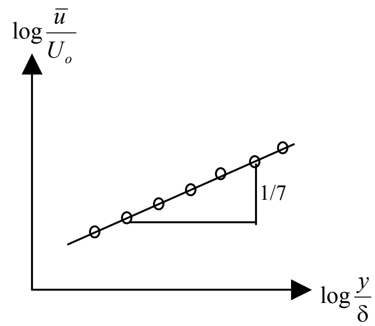
Finally: $\dfrac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \dfrac{\partial \overline{u}_i}{\partial x_j} = \underbrace{-\dfrac{1}{\rho}\dfrac{\partial \rho}{\partial x_i} + \nu \nabla^2 \overline{u}_i}_{\frac{1}{\rho}\frac{\partial}{\partial x_j}\overline{\tau_{ij}}} - \dfrac{\partial}{\partial x_j}\overline{u_i' u_j'}$

$$\left.\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[\overline{\tau_{ij}} - \rho \overline{u_i' u_j'}\right]\right\}$$ **Reynolds averaged N-S eqts. (RANS)**

Reynolds stress $\tau_{R_{ij}} = -\rho \overline{u_i' u_j'}$

4.8 – Turbulent Boundary Layer over a Smooth Flat Plate





(1/7)^th power velocity profile law: $\dfrac{\overline{u}}{U_o} = \left(\dfrac{y}{\delta}\right)^{\frac{1}{7}}$ (1)

where δ = δ(x) to be determined.

$$(1) \rightarrow \delta^* = \frac{\delta}{8}$$

$$\theta = \frac{7}{72}\delta \cong 0.0972\,\delta$$

$$\frac{\tau_o}{\rho U_o^2} = 0.0227\left(\frac{U_o\delta}{\nu}\right)^{-\frac{1}{4}}$$

← Using another empirical formula for friction (Blasius law of friction) for pipes

Von Karman's momentum integral equation:

$$\frac{\tau_o}{\rho U_o^2} = \frac{d}{dx}(\theta) \rightarrow 0.0227\left(\frac{U_o\delta}{\nu}\right)^{-\frac{1}{4}} = \frac{7}{72}\frac{d\delta}{dx} \leftarrow \text{ODE for } \delta$$

$$\frac{\delta}{x} \cong 0.373 R_x^{-\frac{1}{5}}$$

$$\delta(x) \cong 0.373x\left(\frac{U_o x}{\nu}\right)^{-\frac{1}{5}}$$

For δ(0) = 0 and assuming turbulent boundary layer at x = 0, i.e. tripped at x = 0 or $R_x >> 1$

$$\Rightarrow \frac{\delta}{x} \cong 0.373 R_x^{-\frac{1}{5}}$$

$\delta(x) \sim \sqrt{x}$ laminar

$\delta(x) \sim x^{4/5}$ turbulent (grows much faster)

	Laminar	Turbulent
Blasius	$\delta^* \sim 1.72\sqrt{\frac{\nu}{U_o}}\sqrt{x}$	$\delta^* \sim 0.047\left(\frac{\nu}{U_o}\right)^{\frac{1}{5}} x^{4/5}$ .... $\left(\frac{1}{7}\right)^{\text{th}}$ power law

$$D = 0.036\left(\rho U_o^2\right)BL \;\; R_L^{-1/5}$$

$$C_f = \frac{D}{\frac{1}{2}\rho U_o^2 BL} = 0.073 R_L^{-1/5} \quad \text{for } R_L > 5\text{x}10^5$$

<u>Logarithmic Velocity Profile Law</u>

$$\frac{0.242}{\sqrt{C_f}} = \log_{10}\left(R_L C_f\right) \leftarrow \text{Schoenherr's formula}$$