13.021 – Marine Hydrodynamics
Lecture 3

1.7 Stress Tensor

1.7.1 Stress Tensor τ_{ij}

The stress (force per unit area) at a point in a fluid needs nine components to be completely specified, since each component of the stress must be defined not only by the direction in which it acts but also the orientation of the surface upon which it is acting.

The first index i specifies the *direction* in which the stress component acts, and the second index j identifies the orientation of the *surface* upon which it is acting. Therefore, the i^{th} component of the force acting on a surface whose outward normal points in the j^{th} direction is τ_{ij}.

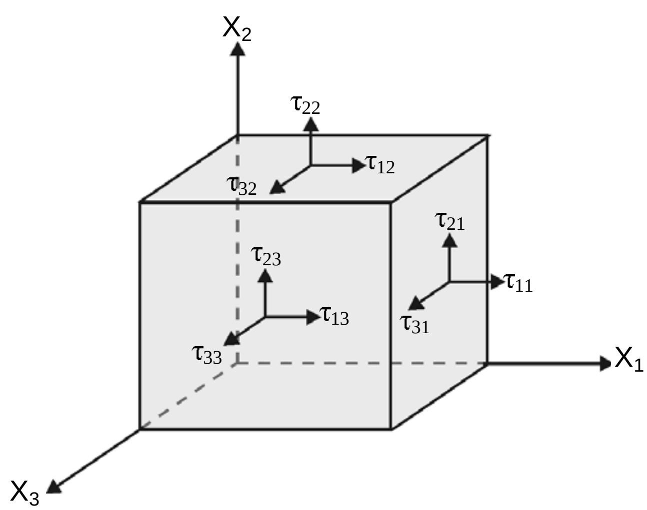

Figure 1: Shear stresses on an infinitesimal cube whose surfaces are parallel to the coordinate system.

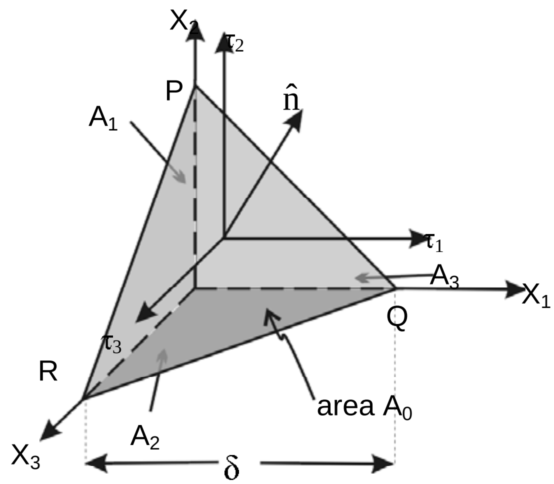

Figure 2: Infinitesimal body with surface PQR that is not perpendicular to any of the Cartesian axis.

Consider an infinitesimal body at rest with a surface PQR that is not perpendicular to any of the Cartesian axis. The unit normal vector to the surface PQR is $\hat{n} = n_1\hat{x}_1 + n_2\hat{x}_2 + n_3\hat{x}_3$. The area of the surface $= A_0$, and the area of each surface perpendicular to X_i is $A_i = A_0 n_i$, for $i = 1, 2, 3$.

Newton's law: $\displaystyle\sum_{\text{on all 4 faces}} F_i = (\text{volume force})_i$ for i = 1, 2, 3

Note: If δ is the typical dimension of the body : surface forces $\sim \delta^2$
: volume forces $\sim \delta^3$

An example of surface forces is the shear force and an example of volumetric forces is the gravity force. At equilibrium, the surface forces and volumetric forces are in balance. As the body gets smaller, the mass of the body goes to zero, which makes the volumetric forces equal to zero and leaving the sum of the surface forces equal zero. So, as $\delta \to 0, \sum_{all4faces} F_i = 0$ for $i = 1, 2, 3$ and $\therefore\ \tau_i A_0 = \tau_{i1}A_1 + \tau_{i2}A_2 + \tau_{i3}A_3 = \tau_{ij}A_j$. But the area of each surface $\perp$ to X_i is $A_i = A_0 n_i$. Therefore $\tau_i A_0 = \tau_{ij}A_j = \tau_{ij}(A_0 n_j)$, where $\tau_{ij}A_j$ is the $\sum$ notation (represents the sum of all components). Thus $\tau_i = \tau_{ij}n_j$ for i = 1, 2, 3, where τ_i is the component of stress in the i^{th} direction on a surface with a normal $\vec{n}$. We call τ_i the stress vector and we call τ_{ij} the stress matrix or tensor.

1.7.2 **Example: Pascal's Law for Hydrostatics**

In a static fluid, the stress vector cannot be different for different directions of the surface normal since there is no preferred direction in the fluid. Therefore, at any point in the fluid, the stress vector must have the same direction as the normal vector $\vec{n}$ and the same magnitude for all directions of $\vec{n}$.

$$\text{Pascal's Law for hydrostatics:} \quad \tau_{ij} = \overbrace{-\,(p_i)\,(\delta_{ij})}^{\text{no summation}}$$

$$\tau = \begin{bmatrix} -p_1 & 0 & 0 \\ 0 & -p_2 & 0 \\ 0 & 0 & -p_3 \end{bmatrix}$$

where p_i is the pressure acting perpendicular to the i^{th} surface. If p_0 is the pressure acting perpendicular to the surface PQR, then $\tau_i = -n_i p_0$, but:

$$\tau_i = \tau_{ij} n_j = -(p_i)\delta_{ij} n_j = -(p_i)(n_i)$$

Therefore $p_o = p_i$, i = 1, 2, 3 and $\vec{n}$ is arbitrary.

1.7.3 **Symmetry of the Stress Tensor**

To prove the symmetry of the stress tensor we follow the steps:

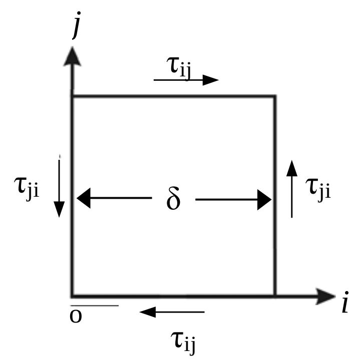

Figure 3: Material element under tangential stress.

1. The $\sum$ of surface forces = body forces + mass$\times$ acceleration. Assume no symmetry. Balance of the forces in the i^{th} direction gives:

$$(\delta)(\tau_{ij})_{TOP} - (\delta)(\tau_{ij})_{BOTTOM} = O(\delta^2),$$

since surface forces are $\sim \delta^2$, where the $O(\delta^2)$ terms include the body forces per unit depth. Then, as $\delta \to 0, (\tau_{ij})_{TOP} = (\tau_{ij})_{BOTTOM}$.

2. The $\sum$ of surface torque = body moment + angular acceleration. Assume no symmetry. Balance of moments about o gives:

$$(\tau_{ji}\delta)\delta - (\tau_{ij}\delta)\delta = O(\delta^3),$$

since the body moment is proportional to δ^3. As $\delta \to 0$, $\tau_{ij} = \tau_{ji}$.

1.8 Mass and Momentum Conservation

Consider a material volume ϑ_m and recall that a material volume is a fixed mass of material. A material volume always encloses the same fluid particles despite a change in size, position, volume or surface area over time.

1.8.1 Mass Conservation

The mass inside the material volume is:

$$M(\vartheta_m) = \iiint\limits_{\vartheta_{m(t)}} \rho d\vartheta$$

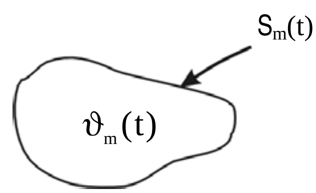

Figure 4: Material volume $\vartheta_m(t)$ with surface $S_m(t)$.

Therefore the time rate of increase of mass inside the material volume is:

$$\boxed{\frac{d}{dt}M(\vartheta_m) = \frac{d}{dt}\iiint\limits_{\vartheta_m(t)} \rho d\vartheta = 0,}$$

which is the integral form of mass conservation for the material volume ϑ_m.

1.8.2 **Momentum Conservation**

The fluid velocity inside the material volume in the i^{th} direction is denoted as u_i. Linear momentum of the material volume in the i^{th} direction is

$$\iiint\limits_{\vartheta_{m(t)}} \rho u_i d\vartheta$$

Newton's law of motion: The time rate of change of momentum of the fluid in the material control volume must equal the sum of all the forces acting on the fluid in that volume. Thus:

$$\begin{aligned}\frac{d}{dt}(\text{momentum})_i =&(\text{body force})_i + (\text{surface force})_i \\ \frac{d}{dt}\iiint\limits_{\vartheta_{m(t)}} \rho u_i d\vartheta =& \iiint\limits_{\vartheta_m(t)} F_i d\vartheta \quad + \quad \iint\limits_{S_m(t)} \underbrace{\tau_{ij} n_j}_{\tau_i} dS\end{aligned}$$

Divergence Theorems For vectors: $$\iiint\limits_{\vartheta} \underbrace{\nabla \cdot \vec{v}}_{\frac{\partial v_j}{\partial x_j}} d\vartheta = \oiint\limits_{S} \underbrace{\vec{v}.\hat{n}}_{v_j n_j} dS$$

For tensors: $$\iiint\limits_{\vartheta} \frac{\partial \tau_{ij}}{\partial x_j} d\vartheta = \oiint\limits_{S} \tau_{ij} n_j dS$$

Using the divergence theorems we obtain

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta_{m(t)}} \rho u_i d\vartheta = \iiint\limits_{\vartheta_{m(t)}} \left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) d\vartheta}$$

which is the integral form of momentum conservation for the material volume ϑ_m.

1.8.3 **Kinematic Transport Theorems**

Consider a flow through some moving control volume $\vartheta(t)$ during a small time interval Δt. Let $f(\vec{x}, t)$ be any (Eulerian) fluid property per unit volume of fluid (e.g. mass, momentum, etc.). Consider the integral $I(t)$:

$$I(t) = \iiint\limits_{\vartheta(t)} f(\vec{x}, t)\, d\vartheta$$

According to the definition of the derivative, we can write

$$\begin{aligned}\frac{d}{dt} I(t) &= \lim_{\Delta t \to 0} \frac{I(t + \Delta t) - I(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \iiint\limits_{\vartheta(t+\Delta t)} f(\vec{x}, t + \Delta t) d\vartheta - \iiint\limits_{\vartheta(t)} f(\vec{x}, t) d\vartheta \right\}\end{aligned}$$

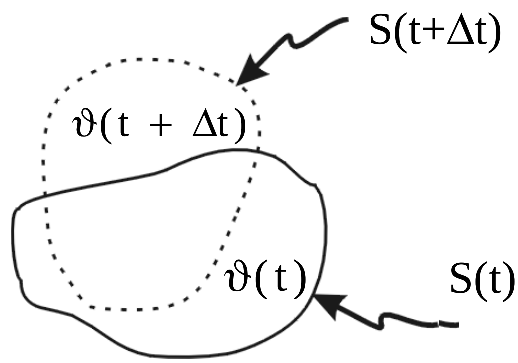

Figure 5: Control volume ϑ and its bounding surface S at instants t and $t + \Delta t$.

Next, we consider the steps

1. Taylor series expansion of $f(\vec{x}, t+\Delta t)$ about $(\vec{x}, t)$.

$$f(\vec{x}, t+\Delta t) = f(\vec{x}, t) + \Delta t \frac{\partial f}{\partial t}(\vec{x}, t) + O((\Delta t)^2)$$

2. $\displaystyle\iiint\limits_{\vartheta(t+\Delta t)} d\vartheta = \iiint\limits_{\vartheta(t)} d\vartheta + \iiint\limits_{\Delta\vartheta} d\vartheta$

   where, $\displaystyle\iiint\limits_{\Delta\vartheta} d\vartheta = \iint\limits_{S(t)} [U_n(\vec{x}, t)\Delta t]\, dS$ and $U_n(\vec{x}, t)$ is the normal velocity of $S(t)$.

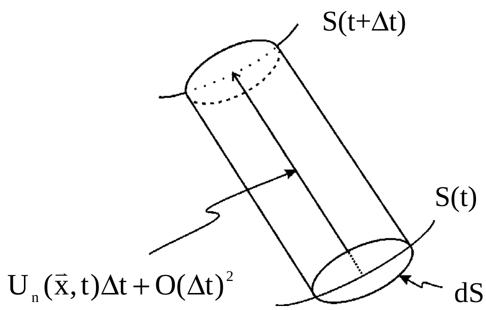

Figure 6: Element of the surface S at instants t and $t + \Delta t$.

Putting everything together:

$$\frac{d}{dt} I(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \iiint\limits_{\vartheta(t)} d\vartheta f + \Delta t \iiint\limits_{\vartheta(t)} d\vartheta \frac{\partial f}{\partial t} + \Delta t \iint\limits_{S(t)} dS U_n f - \iiint\limits_{\vartheta(t)} d\vartheta f + O(\Delta t)^2 \right\} \quad (1)$$

From Equation (1) we obtain the Kinematic Transport Theorem (KTT), which is equivalent to Leibnitz rule in 3D.

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta(t)} f(\vec{x},t)d\vartheta = \iiint\limits_{\vartheta(t)} \frac{\partial f(\vec{x},t)}{\partial t}d\vartheta + \iint\limits_{S(t)} f(\vec{x},t)U_n(\vec{x},t)dS}$$

For the special case that the control volume is a material volume it is $\vartheta(t) = \vartheta_m(t)$ and $U_n = \vec{v}\cdot\hat{n}$, where $\vec{v}$ is the fluid particle velocity. The Kinematic Transport Theorem (KTT), then takes the form

$$\frac{d}{dt}\iiint\limits_{\vartheta_m(t)} f(\vec{x},t)d\vartheta = \iiint\limits_{\vartheta_m(t)} \frac{\partial f(\vec{x},t)}{\partial t}d\vartheta + \iint\limits_{S_m(t)} \underbrace{f(\vec{x},t)(\vec{v}\cdot\hat{n})}_{\substack{f(v_i n_i) \\ \text{(Einstein Notation)}}}dS$$

Using the divergence theorem,

$$\iiint\limits_{\vartheta} \underbrace{\nabla\cdot\vec{\alpha}}_{\frac{\partial}{\partial x_i}\alpha_i}d\vartheta = \oiint\limits_{S} \underbrace{\vec{\alpha}\cdot\hat{n}}_{\alpha_i n_i}dS$$

we obtain the **1^{st} Kinematic Transport Theorem (KTT)**

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta_m(t)} f\left(\vec{x},t\right)d\vartheta = \iiint\limits_{\vartheta_m(t)} \left[\frac{\partial f(\vec{x},t)}{\partial t} + \underbrace{\nabla\cdot\left(f\vec{v}\right)}_{\frac{\partial}{\partial x_i}(f v_i)}\right]d\vartheta,}$$

where f is some fluid property per unit volume.

1.8.4 **Continuity Equation for Incompressible Flow**

- **Differential form of conservation of mass for all fluids** Let the fluid property per unit volume that appears in the 1^{st} KTT be mass per unit volume ($f = \rho$):

$$0 \underset{\substack{\uparrow \\ \text{conservation} \\ \text{of mass}}}{=} \frac{d}{dt} \iiint\limits_{\vartheta_m(t)} \rho d\vartheta \underset{\substack{\uparrow \\ 1^{st}\text{KTT}}}{=} \iiint\limits_{\vartheta_m(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] d\vartheta$$

But since ϑ_m is arbitrary the integrand must be $\equiv 0$ everywhere.

Therefore:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
$$\underbrace{\frac{\partial \rho}{\partial t} + [\vec{v} \cdot \nabla \rho}_{\frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{v}] = 0$$

Leading to the differential form of

Conservation of Mass: $\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0}$

- **Continuity equation $\equiv$ Conservation of mass for incompressible flow** In general it is $\rho = \rho(p, T, \ldots)$, but we consider the special case of an **incompressible flow**, i.e. $\frac{D\rho}{Dt} = 0$ (Lecture 2).

  Note: For a flow to be incompressible, the density of the entire flow need not be constant ($\rho(\vec{x}, t) \neq const$). As an example consider a flow of more than one incompressible fluids, like water and oil, as illustrated in the picture below.

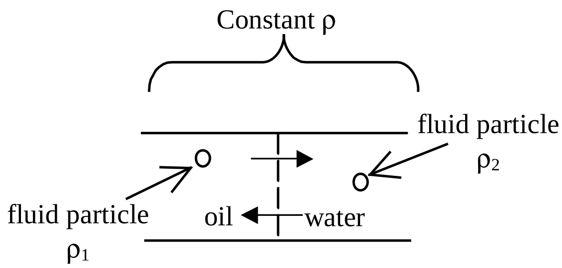

Figure 7: Interface of two fluids (oil-water)

  Since for incompressible flows $\frac{D\rho}{Dt} = 0$, substituting into the differential form of the conservation of mass we obtain the

  Continuity Equation:
$$\boxed{\underbrace{\nabla \cdot \vec{v} \equiv \frac{\partial v_i}{\partial x_i}}_{\text{rate of volume dilatation}} = 0}$$

1.8.5 **Euler's Equation (Differential Form of Conservation of Momentum)**

- **2nd Kinematic Transport Theorem** $\equiv$ 1st KTT + differential form of conservation of mass *for all* fluids. If G = fluid property per unit mass, then ρG = fluid property per unit volume

$$\frac{d}{dt}\iiint\limits_{\vartheta_m(t)} \rho G d\vartheta \underset{1^{st}KTT}{\underset{\uparrow}{=}} \iiint\limits_{\vartheta_m(t)} \left[\frac{\partial}{\partial t}(\rho G) + \nabla\cdot(\rho G\vec{v})\right] d\vartheta$$

$$= \iiint\limits_{\vartheta_m(t)} \left[G \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{v}\right)}_{=0 \text{ from mass conservation}} + \rho \underbrace{\left(\frac{\partial G}{\partial t} + \vec{v}\cdot\nabla G\right)}_{=\frac{DG}{Dt}} \right] d\vartheta$$

The 2nd Kinematic Transport Theorem (KTT) follows:

$$\boxed{\frac{d}{dt}\iiint\limits_{\vartheta_m} \rho G d\vartheta = \iiint\limits_{\vartheta_m} \rho \frac{DG}{Dt} d\vartheta}$$

Note: The 2nd KTT is obtained from the 1st KTT (mathematical identity) and the *only* assumption used is that mass is conserved.

- **Euler's Equation**

  We consider G as the i^{th} momentum per unit mass (v_i). Then,

$$\iiint\limits_{\vartheta_m(t)} \left(F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) d\vartheta \underset{\substack{\uparrow \\ \text{conservation} \\ \text{of momentum}}}{=} \frac{d}{dt} \iiint\limits_{\vartheta_m(t)} \rho v_i d\vartheta \underset{\substack{\uparrow \\ 2^{nd}\text{KTT}}}{=} \iiint\limits_{\vartheta_m(t)} \rho \frac{Dv_i}{Dt} d\vartheta$$

  But $\vartheta_m(t)$ is an arbitrary material volume, therefore the integral identity gives **Euler's equation**:

$$\boxed{\rho \frac{Dv_i}{Dt} \equiv \rho \left(\frac{\partial v_i}{\partial t} + \underbrace{\vec{v} \cdot \nabla v_i}_{v_j \frac{\partial v_i}{\partial x_j}}\right) = F_i + \frac{\partial \tau_{ij}}{\partial x_j}}$$

  And in vector tensor form:

$$\boxed{\rho \frac{D\vec{v}}{Dt} \equiv \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = \vec{F} + \nabla \cdot \ \tau}$$