梁理论

PDF made by Micaimeng

Introduction

Beams have been used since dim antiquity to support loads over empty space, as roof beams supported by thick columns, or as bridges thrown across water, for example. The Egyptians invented the colonnaded building that was the inspiration for the classic Greek temple. Even with the scarcity of timber in Egypt, wooden beams supported the roofs. Early bridges were beams supported at each end by the stream banks, or on piles, on which a deck was constructed for traffic. In either case, the trunk of a tree was the usual beam, trimmed and either left round or squared. Our word "beam" is, in fact, cognate with German Baum or Dutch boom. A tree makes a very satifactory beam, indeed, and practically all beams were originally timber beams. Stone beams, as in door lintels, could be used only for very short spans and light loads, because of the brittleness of stone. Brittle materials do not make good beams.

Through the millennia, beams were designed by empirical methods, applicable only to specific cases and incapable of generalization. Galileo studied beams, and although he did not get it quite right, he showed how the subject should be approached. The theory of beams was only perfected in the late 17th century with the rise of the science of elasticity, and was shown to be a subject of great complexity for which a full and accurate solution was very difficult. This remains true even with modern computational methods, such as the method of finite elements, which produces only numbers (not designs) but very little insight, and depends on parameters that are not well known and models that may contain errors. These methods have great value, but are not a comprehensive solution.

The theory of beams shows remarkably well the power of the approximate methods called "strength of materials methods." These methods depend on the use of statics, superposition and simplifying assumptions that turn out to be very close to the truth. They give approximate, not exact, results that are usually more than adequate for engineering work. Calculus and a little differential equations are all the mathematics required for this approach, not the partial differential equations or tensor analysis that are typical tools in elasticity.

Strength of materials methods can be used for beams of arbitrary cross sections, for beams whose shape varies along the length, for loads applied in any direction at any point, distributed or concentrated. Many of these applications are discussed in the first reference, which shows the versatility of the method. The results obtained are fully adequate for engineering design. On the other hand, an accurate and rigorous quantitative solution in these varied cases would be extremely difficult and usually impossible.

An introduction to many of the concepts that will be needed here will be found in Elasticity, including the meaning of shear and bending moments, and shear and moment diagrams.

Pure Bending

A beam is in pure uniform bending when the shear stress in the beam is zero, and the bending moment is constant. It is not very easy to achieve this state in practice.

Opposite couples of moment M applied to the ends of a uniform beam creates pure bending, and there must be no transverse loads. If the ends of a beam are joined by a cord in tension, as in an archery bow, the beam is in pure bending with a superimposed axial compression. In the strength of materials picture, we would consider this as the superposition of uniform bending and uniform compression, which would be treated separately. Let us assume here that a beam under consideration has a cross section symmetrical with respect to a plane that is normal to the bending moment. Deflections will be in this plane, and we will establish x and y axes such that the x-axis is along the beam, and y is either upwards or downwards. Usually, y is taken positive downwards, and then the positive z-axis is into the plane, and a positive moment is clockwise.

In strength of materials, we assume that the curve assumed by a beam in pure uniform bending is circular. Transverse planes remain plane, and intersect at an axis parallel to the z-axis and a distance R above the reference line defining the axis of the beam. This reference line can be defined rather arbitrarily, so we shall take it as the line through the centroid of the cross section, which will turn out to be significant. R is the radius of curvature of the stressed beam, and its curvature is $\kappa = 1/R$. An axial distance L before bending at a position y changes in length by $\Delta L = (L/R)y$, so the longitudinal strain is $\Delta L/L = y/R = \kappa y$. This simple assumption proves to be very close to the fact in most cases. It is an important conclusion that the plane of the cross-section will not warp, but remain plane. Because lateral strains are related to longitudinal stresses, the cross-section will change slightly in shape, however.

The longitudinal stress σ will be proportional to the strain. Since the beam is not constrained laterally, $\sigma = Y\varepsilon = \kappa Yy$, and Y is the Young's Modulus, with the same dimensions as the stress. Statics requires that the net force on a cross section of the beam be zero (in a free-body diagram of, say, the portion of the beam to the right of the cross-section there are no other longitudinal forces). This means that $\kappa Y \int ydA = 0$, or $\int ydA = 0$. This is precisely the condition that y = 0 locate the centroid of the cross-sectional area. The axis y = 0 is, then, called the neutral axis because the longitudinal stresses there are zero.

Since the normal forces are opposite for y > 0 and Y < 0, they most certainly will exert a moment $\kappa Y \int y2dA$ in the z-direction, and this must equal the applied bending moment M for equilibrium. A free-body diagram of the portion of the beam to the right of the plane considered is shown in the figure. The integral is called the moment of inertia of area, and is represented by I. Then, we have $\kappa YI = M$, or $\kappa = M/YI$. We have now found the curvature of the beam in terms of the applied bending moment, which is a rather exciting result, and one which Galileo would have admired.

It is almost anticlimactic to be able to determine the stresses, since we have immediately that $\sigma = \kappa Yy = My/I$. Remarkably, the elastic modulus Y has cancelled out. If c is the maximum distance from the neutral axis to the top or bottom of the beam, $\sigma max = M(c/I) = M/S$, where $S = I/c$ is called the section modulus, even though it is not a modulus at all, having the wrong dimensions. If we know the maximum design stress and the bending moment, then we can find the required S and use it as a guide to the selection of the beam. This simple process works very well for steel or wooden beams, which will usually fail from exceeding a maximum stress before anything else bad happens.

Two useful examples, especially for wooden beams, are the rectangular beam of height h and width b, and the circular beam of diameter d. The moment of inertia of area of the rectangular beam about a centroidal axis parallel to the width is I = bh3/12, and for a circular beam, I = πd4/64. The corresponding section moduli are S = bh2/6 and S = πd3/32. If we know the design bending moment M and the working stress σ, then S = M/σ, and from this value we can select the beam. The diameter of a circular beam is given at once; for the rectangular beam, we must decide on a ratio of the depth to the width, on the basis of other considerations. This is not all there is to beam design, but it is a major step.

Shear

Under more general loading conditions, a transverse force will act on the cut surface of the free body we considered above. The shear is negative of the sum of the forces on the beam to the left of the section. Just as we considered a bending moment only in one plane for simplicity, the shear forces will be considered to act vertically only. This vertical force is distributed over the section, and its average value is V/A, where V is the shear, positive downward (in the direction of increasing y). The distribution is, however, by no means uniform, so we need to know how the shear stress is distributed.

Shear stress has the peculiarity that it is in opposite directions on two parallel bounding surfaces (so that the net force will be zero), and even more importantly, there must be shear stresses of equal amounts at right angles (so that the net moment will be zero). Instead of finding the vertical shear stress at some height y, it will be easier to find the horizontal shear stress at that height. Once we have done so, it will be equal to the vertical shear stress.

To find the shear stress, we consider the shaded portion of the beam shown in the diagram as a free body. On the lower surface, the stress is zero. On the end faces, the stresses in the x-direction are due to the bending moment, and are proportional to the bending moment M and the distance from the neutral axis y. In fact, σ = My/I. If V > 0, then M' is greater than M by Vdx, and the total force on the right-hand face will be larger than the total force on the left-hand face. The difference must be balanced by the shear stresses over the upper face, which give a total force of τtdx, where dx is the length of the element, and t is its width. We assume that the shear stress τ is constant across the width of the element, which it is if the sides are parallel where the stress is being found. The result of this summation of forces is that τ = VQ/It, where Q is the first moment of the shaded area with respect to the neutral axis. This is easily found if the shaded area is a rectangle, since then it is just the area of the rectangle times the distance from its centroid to the neutral axis. Everything needed for the derivation is found in the diagram, which will repay close study. Even when we must make different assumptions, the calculation of shear stresses is based on the same principles.

Statics gives us the value of the shear V at any cross-section. We can find the average shear stress quite simply, as V/A, where A is the area of the cross-section. If we consider the rectangular and circular beams, the area moment Q in the shear formula is easy to evaluate. If y is the distance from the neutral axis, then for the rectangular beam Q = (b/2)(h2/4 - y2). The maximum Q clearly occurs for y = 0, and equals bh2/8. Therefore, the maximum shear stress is τ = V(bh2/8)/(b2h3/12) = (3/2)(V/bh), that is, 1.5 times the average shear stress. This stress may have two effects: it may split the beam longitudinally at the neutral axis, or it may cause the beam to buckle in shear. For a rectangular beam, the latter outcome is improbable, but the first could happen. If τ is the working stress in shear, then bh > 3V/2τ to guard against this. This is the other condition necessary in the design of a rectangular wooden beam--it must be wide enough to resist the shear stress. To resist the bending, we have already found that bh2 > 6M/σ, so we get h = 2Mτ/σV.

The shear formula isn't strictly applicable to the circular beam, but may be assumed to apply at the diameter of the beam, where Q = d3/12. Then τ = V(d3/12)/(πd4/64) = (4/3)(V/A). After we have found a diameter that will resist the bending, the shear stress should be checked to see that it is permissible. In most cases it will be, because the circular beam's shape encourages a low maximum shear stress.

Reasonable working stresses for wood are 1000 psi in compression and tension, and 70 psi in shear parallel to the grain. Consider a circular beam 12" in diameter. Since the section modulus is 170 in3, it will resist a bending moment of 170,000 lb-in or 14,200 lb-ft. If the beam is 30 ft high, and a force acts at the end of it, the force will be 473 lb, and this will be the shear in the beam. The maximum shear stress will be about 5.6 psi, well below the limit. If the beam is only 4 ft high, the force will be 3550 lb, so the shear stress will be 42 psi, still well below the limit.

Now consider a rectangular beam with a 4 ft span, loaded with a concentrated force P lb at the center. The maximum bending moment will be 12P lb-in. If the beam is a 2x4 (1.5" x 3.5"), the area is 5.25 in2, the moment of inertia of area is 5.36 in4, and the section modulus is 3.06 in3. Therefore, 12P = (3.06)(1000), or P = 255 lb. Since the shear is half this, or 128 lb, the maximum shear stress will be τ = (3/2)(128/5.25) = 37 psi, well within the working stress limit. If we used a 1x6 (0.75" x 5.5") instead, then the area would be 4.125 in2, the moment of inertia of area 10.4 in4, and the section modulus 3.78 in3. In this case, P = 315 lb, and τ = 58 psi;, which is approaching the limit. If the span were only 2 ft, then P = 630 lb, but τ = 115 psi, so the beam would be in danger of splitting lengthwise at the maximum load.

Rolled metal beams have their areas concentrated in upper and lower flanges, connected by a thin web. The flanges resist the longitudinal tensile and compressive stresses, while the web resists the shear. Consider an idealized shape consisting of two areas A separated a distance h apart by a web of thickness t. If this shape resists a bending moment M, then the force F in the flanges is F = M/h, and so the stress there is σ = M/Ah, compressive in the upper flange and tensile in the lower. The effective section modulus is S = Ah, and the moment of inertia of area is Ah2/2. If we assume that the web does not resist any of the bending moment, and the flanges none of the shear, then Q = 2A(h/2) = Ah for any y, so the shear stress will be constant in the web. This stress is τ = VAh/(Ah2/2)t = V/ht, which is the average stress, of course. These simple formulas will go a long way in designing a beam with an appropriate shape.

This shear stress acts between the web and the flanges. In a solid beam, it is resisted by the solid metal and must not be greater than the allowable shear stress. Actually, it is seldom a problem in this case. In a built-up beam, this stress must be considered in designing the connections betweeen the web and the flanges. If the web is thin, shear stress may cause buckling, in which the web wrinkles, fails to support the flanges, and the beam fails. To prevent this, the web may be strengthened by stiffeners, vertical members connected at intervals with the web. The allowable shear stress depends on the ratios of the stiffener spacing to the web depth, and of the web depth to its thickness. Values are given in handbooks.

Deflections

A beam that is initially straight will deflect under load. In the problems that are of most interest to us, the deflections will be small compared to the span of the beam, and the angles of deflection will by less than one degree in most cases. In that case, we can always replace sin θ and tan θ by θ itself (in radians, of course), and cos θ by 1. The curvature of the beam will be given well enough by the second derivative of the curve y(x) of the deflected beam. From our theory of bending, then, we have that y" = κ = M/YI (with y positive upwards). If M = M(x) and Y, I are constant, this equation is very easily integrated to give y(x).

The differential equation for the elastic curve can also be expressed in terms of the shear V or the distributed load q, simply by differentiating the above equation. We find that y"' = V/YI and y"" = -q/YI. These may sometimes be easier to use, but they involve more integrations and more evaluations of constants than the moment equation.

Let's do a simple example to show how this works. Consider a simple beam with a concentrated load P at the center. In this case, M(x) = Px/2, so we have y" = (P/2YI)x. Integrating once, y' = (P/4YI)x2 + C. Since y' = 0 at x = L/2, we have C = -(PL2/16YI). Integrating again, y = (P/12YI)x3 - (PL2/16YI)x + C'. Since y = 0 at x = 0, C' = 0. The deflection δ = -y = (P/4YI)(x2/3 - L2/4)x, where x is measured from the left-hand end toward the center. Deflections on the other half of the beam are symmetrical. The maximum deflection occurs at the center, and is δ = PL3/48YI. We note that it is proportional to the load P, to the cube of the span, and inversely proportional to the rigidity YI.

There is a very pretty way of finding deflections that is semi-geometrical and based directly on the equation y" = M/YI. In its simple form, it depends on the smallness of the angles of deflection. y" can be written dy'/dx, or d(tan θ)/dx = dθ/dx, approximately, where θ is the angular rotation of an element dx of the beam axis. This means that the difference in angles at the two ends of a segment AB is simply the area under the curve of M/YI. Suppose we consider an element dx, and we want the deflection at point B at x', a distance (x' - x) from dx. The contribution dx to the deflection is the difference in angles times the distance to B, or (x' - x)(M/YI)dx. The total contribution from AB will be the integral ∫(A,B)(x' - x)(M/YI)dx, which is the moment of the area under the M/YI curve from A to B with respect to B. It is, in fact, the deflection of B from the tangent at A. This is rather confusing to read, but some examples should make it clear.

The diagram at the left shows a cantilever beam of length L with a concentrated load P at the end. The moment diagram is a triangle, as is the M/YI diagram, since we assume that Y and I are constant. The tangents at the ends of a small interval dx are shown, and their contribution to the deflection of the end of the beam. The difference in angles of the tangents is the dark shaded area under the M/YI curve. The deflection is the moment of the total shaded area about the end of the beam. The distance from the centroid of the triangular area to that point is 2L/3, so the result is as shown. In this example, the tangent at A was horizontal.

The diagram at the right shows a simple beam of length L with a concentrated load at the center. The tangent at A is not horizontal, but slopes downward at an angle equal to the shaded area of the M/YI curve, since the slope at the center is zero. The difference δ' is given by the moment of this area about the center of the beam, which is its area times L/6, as shown. The deflection δ is the full distance to the tangent at A, less δ', as shown. This is called the moment area method for finding deflections.

Since the (approximate) equations for deflection are linear, deflections can be superposed. The use of this principle is rather obvious, but there is an interesting application. If we know the deflection at some point of a beam caused by a unit load anywhere on the beam, as a function d(x), then by considering a distributed load w(x) as a superposition of concentrated loads w(x)dx, we can find the deflection at the point under consideration as the integral δ = ∫ w(x)d(x)dx. For example, the deflection at the end of a cantilever of length L caused by a unit load at x is x2(3L - x)/6YI. Under a uniform load of w per unit length, the deflection at the end is obtained by multiplying by w and integrating from 0 to L. The result is wL4/8YI.

The deflection of a beam when a load P is applied reversibly implies that energy is stored in the stressed, deflected beam. Since the deflection is proportional to the load, or P = Kδ, the energy stored is Pδ/2 or Kδ2/2, a familiar result. K is the stiffness or spring modulus. There is a similar relation between bending moment and angle, in which M = Gθ. The energy is Mθ/2 = M2/2G in this case. For a length dx, we know that dθ = (M/YI)dx, so the bending energy in length dx of the beam will be u dx = (M2/2YI) dx. In most cases, this will be the largest part of the strain energy of a beam, since the energy due to shear deformations will be small.

The moment in a cantilever beam of length L when a concentrated load P acts at its end is M = Px, where x is the distance from the end of the beam. The total strain energy of the beam is then U = ∫ (P2/2YI)x2dx = P2L3/6YI. This must be equal to the work done in applying the load, Pδ/2, so we find that δ = PL3/3YI, once again.

Suppose that we drop a weight W from a height h onto a beam. For concreteness, suppose this is the end of a cantilever of length L. When the accelerated mass contacts the beam, a decelerating force is applied proportional to the deflection of the beam, which brings the weight W to rest at some deflection δ, which is, of course, greater than the static deflection under the load W. We can estimate this deflection by equating the total energy of the mass to the strain energy of the beam, so that W(h + δ) = U. Let the static deflection under the weight W be δ' = WL3/3YI. Then, we get the equation δ2/2δ' - δ - h = 0, a quadratic equation for δ. Its solution is δ = δ' + √(δ'2 + 2hδ'). If h >> δ', then δ = √(2hδ'). On the other hand, if h = 0, so that the weight W is released when it is in contact with the beam, then δ = 2δ', and the deflection is twice the static deflection.

There are lots of approximations involved here. When the falling weight strikes the beam, the beam has to be accelerated, so there is some impact and loss of energy on this account. The depression of the beam brings in a little additional gravitational energy, and the kinetic energy of the beam is recovered at maximum deflection. Subsequently, the weight is projected upwards, while the beam vibrates. It is a very complicated process in which our approximations have made it possible to see the main result without having to consider all the small effects that are certainly there, but are negligible. This is another example of the genius of the strength of materials approach.

We also learn that a suddenly applied load, even if applied without shock, is equivalent to twice the static load. If the load is applied impulsively, there are even greater effects. This must always be taken into account with bridge traffic loads, which are at best suddenly applied. A bridge that will take moving loads is hardly bothered by the equivalent static loads. This is a lesson that it took bridge designers many years to learn (along with the effects of stress concentrations and fatigue).

Columns

The theory of beams even has an important and interesting application to the theory of columns, the other basic structural element. A column is an element supporting a load by axial compression. A short column, or strut, fails by deforming and crushing when its compressive strength is exceeded. A long, slender column fails in a totally different way, by deflecting to the side and collapsing, or buckling, without ever exceeding its elastic limit until the deflection becomes large and collapse is inevitable. Intermediate columns buckle by inelastic deformation in a very complex way. We won't consider the complete theory of columns here, which is difficult and involves empirical measurements, but only the theory of slender columns that comes from the theory of beams.

Consider a column to be a beam on edge, vertical instead of horizontal. Unlike a beam, a column necessarily has a longitudinal stress. However, when transverse forces and moments act upon it, it will deflect to the side just as a beam deflects vertically. As long as all stresses are within the elastic limit, all our assumptions about superposition and small displacements will apply here as well. Any column for which this is the case is called a slender column.

Consider, then, a column constrained as shown in the diagram. Its lowest point is pin-connected, which means that it may rotate freely but not translate. The highest point, on which the vertical load P acts, is also pin-connected, but may translate vertically. If the column deflects to the right through a distance y(x), then every point is subjected to a bending moment Py. The differential equation for the deflection is YIy" = - Py, or y" + k2y = 0, with k2 = P/YI. This is a very familiar equation whose solution is y = A sin kx + B cos kx. The boundary conditions are y = 0 at x = 0 and x = L, so B = 0 and sin kL = 0, while A is arbitrary.

Since sin kL = 0, kL = nπ. Therefore, the solution y = A sin π(x/L) has the smallest value of k, and the deflection is a sinusoidal arch. Then, P/YI = π2/L2, or P = π2YI/L2. If P is even slightly less than this value, the column will straighten itself elastically, since then A = 0, and so y = 0, is the only possible solution satisfying the boundary conditions. If P is slightly greater than this amount, any chance displacement making A ≠ 0 will grow without bounds, and the column will collapse. Therefore, P is the critical load for buckling. Since P is even larger when n ≠ 0, it is easy to see why we throw these solutions out.

This analysis was first made by Euler in 1744, and though at first ignored by practical builders, is still the basis of column design today. It must always be remembered that it applies only to a slender column, and to the initiation of buckling, if its predictions are to be valid. Nevertheless, the theoretical buckling load is a useful parameter in column design.

The moment of inertia of area can be written I = Ar2, where r is the radius of gyration of the area. Then, P/A = π2Y/(L/r)2. P/A is an average stress (it is the actual stress only when the column is vertical and undeflected), and L/r is the slenderness ratio of the column. For a column to be slender, and for Euler's theory to apply, P/A must be less than the proportional limit for the material, which can be taken as the yield strength σy. For a yield strength of 36 ksi (typical of structural steel), L/r is about 91.

The buckling load does not depend on the strength of the column, only on the modulus of elasticity of the material and the moment of inertia of area of the cross-section, in addition to the length. Strength does not enter until after the column has buckled, and then it only affects the details of the collapse.

Pinned ends reflect the tendency of connections to rotate, especially through small angles, and is a conservative assumption. We can find the buckling load for certain other end conditions by inspection, so long as the inflection points of the sinusoid can be located. For example, if the column is built-in, or fixed, at both ends, there must be a whole wavelength in the distance L, so that a half-wavelength is L' = L/2, with distances of L'/2 at each end, with y = 0 at the quarter points. Then, P = 4π2YI/L2. The buckling load is increased by a factor of 4 by building-in the ends.

If the lower end is built-in, but the upper end is free to move sideways, then L represents a half-wavelength, and L' = 2L. Now P = π2YI/4L2. The buckling load is decreased by a factor of 4. The length L' is called the effective length, and may be expressed as L' = KL. We have just found that K = 1/2 for fixed ends, and K = 2 for the upper end free. Slenderness ratios for use in the formula are, then, KL/r. r is always the least radius of gyration with respect to any axis normal to the axis of the compression member. Most codes give maximum allowable slenderness ratios, even for tension members. A ratio over 200 is usually prohibited, since such a member would turn out to be a wiggly-wobbler.

An important case is that of a fixed, or built-in, lower end, but with the upper end pinned and allowed to move vertically, as shown in the diagram. Now we have a moment arising at the fixed end to hold it vertically in equilibrium. If R is the sideways component of the reaction at the upper pin, this moment is RL from statics. Now the moment in the column is Py - R(L - x), so y" + (P/YI)y = (R/YI)(L-x) is now the differential equation for the deflection. The general solution of this equation is y = A sin kx + B cos kx +(R/P)(L - x), as substitution in the equation will confirm. The boundary conditions are y(0) = y(L) = 0, y'(0) = 0. These require that B = -RL/P, A = R/Pk, and tan kL = -B/A = kL.

The equation for kL is a familiar transcendental equation, with smallest root kL = 4.4934 = 1.4303π (easily found with an HP-48). The deflection is y = (R/Pk)[sin kx - kL(cos kx - 1) -kx]. R/Pk is just an arbitrary constant, as in the case of the pinned-pinned column. The buckling load is given by P = (1.4043 π)2YI/L2 = 2.046 π 2YI/L2. In this case, L' = L/ √(2.046) = 0.699L. The buckling load is approximately doubled by building in the lower end.

The strongest column for a given cross-sectional area has the largest moment of inertia of area, which means that the area is as far from the neutral axis as possible. The strongest column shape is something of a surprise, since it is not a circle. If we express the moment of inertia of area in terms of the area, then for a circular section I = 0.0796A2, for a square section I = 0.0833A2 (about an axis parallel to a side, or corner-to-corner), and for an equilateral triangle, I = 0.0888A2, about an axis parallel to a side. The moment of inertia about an axis through a vertex is 0.9622A2, which is larger (and erroneously used by Gere and Timoshenko in their example). The equilateral triangle actually makes the strongest column for a given area, but not by much (12% stronger than the circle). A column will fail about the axis of smallest moment of inertia of area.

The most efficient column will quite obviously be a hollow shape. A thin-walled pipe of wall thickness t and diameter d has a radius of gyration of about d/2 √2. Of course, the wall must be thick enough to resist buckling on a smaller scale. Consider an aluminium beer can as an example. With a length of 120 mm and a diameter of 65 mm, L/r = 5.2, so it will act like a strut. The aluminium is about 0.1 mm thick, so A = 20.4 mm2 or 0.0316 in2. If the yield strength is 50 ksi (it varies very widely for different alloys), then the beer can could resist a load of 158 lb. The can will, in fact, fail under a much smaller load if it is twisted so that the walls buckle. The thin walls have a much smaller radius of gyration against buckling. The strength of curved plates is remarkable, but buckling always lurks in the background.

Square pipes are a little more efficient than circular pipes, for the same reason that a solid square column is more efficient than a solid circular column, and square steel pipes are manufactured for use as columns.

Buckling in the compressive flange of a beam is analyzed in a similar way. The radius of gyration that is used corresponds to the area of the flange plus one-third the height of the web.

We probably should not leave this subject without considering the case when the axial load is applied eccentrically to pinned-pinned beam, at a distance e from the axis. The result is simply to add Pe to the moment bending the column, so that the differential equation becomes y" + k2y = -k2e. All we have to do is to subtract e from the deflection y. However, now we can evaluate all the constants from the boundary conditions and find that y = e[tan(kL/2)sin kx + cos kx - 1]. This allows us to find the maximum deflection δ = e[sec(kL/2) - 1], and the maximum bending moment M = Pe sec(kL/2). This is typical of actual behavior, even when the eccentricity e cannot be determined accurately, because of various imperfections in a beam and its connections.

This solution, since it gives a definite deflection, permits us to estimate the ultimate-fibre stress σ as P/A + M/S, superposing the effects of axial stress and bending. The result for the maximum stress can be put into the form σ = (P/A){1 + (ec/r2) sec [(L/2r) √ (P/YA)]}, called the secant formula. c is the distance from the axis of bending to the ultimate fibre. It gives an allowable stress P/A for a given section if we assume a certain maximum stress σ and an eccentricity e. It is often assumed in steel construction that ec/r2 = 0.25. The secant equation, which requires an iterative solution, can be easily solved by the HP-48. The secant equation is the basis for many column design procedures. With an assumed value of ec/r2, it is an alternative to the empirical column formulas discussed below. However, it does not appear in the AISC code, and does not seem to be favored at present.

Most columns turn out to be intermediate columns in the region of inelastic buckling. Good results have been obtained by assuming that the allowable stress is given by σ = σy - A(L/r)2, where σy is the yield stress and A is a constant, for small L/r, and by the Euler expression π2Y/(L/r)2 for large L/r. These two expressions are made to join smoothly at some boundary value of L/r = r'. By matching value and slope, r'2 = 2π2Y/σ2, and A = π2Y/r'4. This is called a parabolic formula, which appears in many forms in handbooks and codes. A simple example is σ = 36,000 - 1.1(L/r)2 for L/r < 128, and 2.96 x 108/(L/r)2 for L/r > 128, which can be used for A36 steel, and does not include a safety factor.

The parabolic formula is one of several empirical equations that bridge the gap between short and long columns. As shown in the diagram, these begin at the yield stress σy, which may be taken as an ultimate or a working stress and intersect the Euler curve at some boundary value of L/r. In the simplest case, the working stress is constant at the yield stress up to the point a, where it intersects the Euler curve at L/r = √ (π 2Y/ σ y). This relation is not generally recommended. A linear formula results when a straight line from the yield stress at L/r = 0 intersects the Euler curve at a point c, where it is tangent and L/r = √(3π2Y/σy). This formula was proposed by T. H. Johnson in 1886, and was used for many years in the U.S., until it was superseded by the parabolic formula. The parabolic curve intersects the Euler curve at point b, where L/r = √(2π2Y/σy). The theoretical π2 can be replaced by an empirical constant (as high as 15) or adjusted to give working stresses. The forms of the various formulas are quite clear, and should not be daunting. Of course, any formula used in practice should be backed up by experiment. The various boundary values of L/r are used to classify columns as long or short. It can be seen that there is no rigid boundaries, but the range of L/r for which the formula is valid should be given.

Another kind of empirical formula is the Rankine type, of the form σ = A / [1 + (L/r)2/B]. A should be the stress for L/r = 0, and AB should be the proper constant for Euler's formula. A Rankine formula used by the AISC had A = 18,000 psi and B = 18,000, and valid for 120 < L/r < 200. This formula gave the safe load. Rankine formulas also give a smooth curve, and are nearly equivalent to the parabolic formulas. I have discussed quite a few kinds of column formulas here, since they may appear very confusing in code specifications, but are really all based on simple principles.

The AISC defines Cc = √(2πY2/σy), which is r', the boundary slenderness ratio. For A36 steel, it is about 128. Then, for L/r > Cc, the allowable stress is σ = (12/23)[π2Y/(L/r)2]. For smaller L/r, the allowable stress is [1 - (L/r)2/2Cc2]σy divided by 5/3 + 3(L/r)/8Cc - (L/r)3/8Cc3. These two expressions join smoothly at L/r = Cc. This is clearly a formula of parabolic type with a somewhat more elaborate behavior for small L/r, substantiated by test. It includes a safety factor, so it give working stresses. The AISC uses KL/r instead of L/r, where K is the factor depending on end conditions that was discussed above. This allows the designer to include the effects of different end constraints.

Materials

A popular material for steel construction is ASTM A36 low-carbon structural steel, which replaced the venerable ASTM A7 structural steel in the 1970's. The yield strength Fy of this steel is 36 ksi (36,000 psi), and its ultimate strength is about 60 ksi. It is quite ductile, with a 30% elongation in the tensile test. It is a very reliable material because of its ductility, and can be designed on a plastic basis as well as on the working-stress method used here. There is always a tendency to use higher-strength steels for reasons of economy (lower dead load), but these steels are more brittle and less reliable than A36 steel. A36 steel is similar to an SAE 1015 steel, with less than 0.2% carbon. The usual working stress in bending is taken as 0.66Fy, or 24 ksi. This gives a factor of safety of 2.5 against the ultimate strength, while ensuring that service deflections will remain in the elastic region.

The widely-used W, or wide-flange, sections are very close to the ideal beam consisting of compression and tension resisting flanges separated by a shear-resisting web. The surfaces of the flanges and web are parallel, joined by fillets to discourage stress concentration in the re-entrant corners. The "I-beam" is an S section, with the inner faces of the flanges inclined at a 1:6 angle. Channel, or C sections, also have inclined flanges. M and MC shapes are special ones that do not fit in the other categories. Rolled sections have large residual stresses, which do not have deleterious effects thanks to the plasticity of the steel.

The compressive flange of a beam acts as a column, and may buckle to one side or the other when loaded too heavily, as mentioned above. This catastrophic mode of failure limits the span of the beam between lateral braces that prevent the flange from deflecting to the right or left. For any section and working stress, there is a maximum unsupported span. Handbooks give these lengths for specific sections and working stresses. If the working stress is limited to 0.6Fy, a longer unsupported span can be used than for the usual working stress.

These matters can be illustrated by considering a W10 x 49 section, which is 10" high (the first number) and weighs 49 lb/ft (the second number). This section is 10" high and 10" broad. The flanges are 9/16" thick and the web is 5/16" thick. This section can be considered approximately as two flanges of area 5.625 in2 held 9.44 in apart by the web. We can calculate the properties of the section in this approximation as follows (accurate values in parentheses): area, 14.2 in2 (14.4); moment of inertia of area, 251 in4 (273); section modulus 53.1 in3 (54.6). We see that the approximate values are pretty close, and are conservative estimates.

Suppose that the beam has a span of 20 ft (240 in), and supports a concentrated load P at its center. The maximum bending moment is 60P lb-in. Then, (24,000)(53.1) = (60P), or P = 21,200 lb = 21.2 kip. The bending moment is 106 kip-ft, and the shear is 10.6 kip. The shear in the web is then 10.6/ht = 2.95 ksi. The AISC manual (p. 2-10, 7th ed.) gives the maximum unsupported length for this working stress as 10.6 ft. Therefore, a lateral brace is required at mid-span. If we had used a working stress of 0.6Fy or 22 ksi, then the unsupported length could be 25.9 ft, and no bracing would be required. The load P would have to be reduced to 19.5 kip, however.

As for web buckling from shear, the ratio h/t = 30, which is smaller than the tabulated values in the AISC manual (p. 2-133, 7th ed.), showing that there is no danger of web buckling in this case, which is usual for solid sections. Web crippling is a distinct effect that occurs near concentrated loads or supports, where the web buckles under the localized compressive load.

We also note from the AISC handbook that the lightest section that could support this load would be the W18 x 35. In fact, P could be raised to 23.2 kip with this section. However, the maximum unsupported length of flange would be only 6.3 ft, so we would need four lateral braces.